IMAGE AND SIGNAL RESTORATION USING PAIRWISE MARKOV TREES

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ABSTRACT

This work deals with the statistical restoration of a hidden signal using Pairwise Markov Trees (PMT). The latter PMT, recently introduced in the case of discrete hidden signal, are here applied to unsupervised image segmentation and it is showed that they work better then the classical Hidden Markov Trees (HMT). Further, considering a PMT in a linear Gaussian model with continuous hidden data, which is new, we give the formulas of an original extension of the classical Kalman filter.

1. INTRODUCTION

Hidden Markov models (HMM), like hidden Markov chains (HMC), hidden Markov fields (HMF), or hidden Markov trees (HMT) admit numerous applications in various domains, and in particular in signal and image processing. These models have been recently generalized to pairwise Markov chains (PMC [10]), pairwise Markov fields (PMF [6]), and pairwise Markov trees (PMT [7]) The aim of this paper is to present some further properties of the PMT introduced in [7] (in French). On the one hand, we present an application to unsupervised image segmentation of the PMT with discrete hidden process. On the other hand, we propose an original extension of the well known Kalman filter to the PMT with continuous hidden process. The latter extends analogous results proposed in the case of PMC [2, 9].

2. HIDDEN, PAIRWISE, AND TRIPLET MARKOV TREE

Let S be a finite set of points and $X = (X_s)_{s\in S}$, $Y = (Y_s)_{s\in S}$ two stochastic processes indexed on S. Each X_s takes its values in Ω (which will be a finite set in the next section and R^N in section 3) and Y_s takes its values in the set of observations, which will be real numbers R in the next section and R^q in section 3. Let S^1, …, S^n be a partition of S representing different « generations ». Each $s \in S^i$ admits $s^+ \subset S^{i+1}$ (called his « children ») in such a way that every element of $t \in S^{i+1}$ has a unique « parent » $t^- \in S^i$. We assume that S^1 is a singleton (its element s_r is called « root »). Then the distribution $p(x, y)$ of (X, Y) can be defined by four models with increasing generality :

(i) the classical Hidden Markov Tree with independent noise (HMT-IN [1, 4]), in which $p(x) = = p(x_{s_r}) \prod_{s\in S-S^1} p(x_s | x_{s^-})$ (which means that X is a Markov tree), $p(y|x) = \prod_{s\in S} p(y_s|x_s)$, and thus, putting $z = (x, y)$:

$$p(z) = p(x_{s_r}) p(y_{s_r}|x_{s_r}) \prod_{s\in S-S^1} p(x_s|x_{s^-}) p(y_s|x_s) ; \qquad (1)$$

(ii) the Hidden Markov Tree (HMT), in which X is a Markov tree as above and the pairwise process $Z = (Z_s)_{s\in S}$, where $Z_s = (X_s, Y_s)$, is a Pairwise Markov Tree (PMT), which means that its distribution verifies :

$$p(z) = p(z_{s_r}) \prod_{s\in S-S^1} p(z_s|z_{s^-}) ; \qquad (2)$$

(iii) the PMT $Z = (Z_s)_{s\in S}$ verifying (2);

(iv) the Triplet Markov Trees (TMT [8]), in which one introduces a latent variable $U = (U_s)_{s\in S}$ and assumes that the triplet $T = (X, U, Y)$ is a Markov tree (i.e., verifies (2) with $t = (x, u, y)$ instead of $z = (x, y)$).

Let us remark that the greater generality of PMC with respect to HMT-IN appears locally at the transition probability level. In fact, as $p(z_s | z_{s^-})$ in (2) can be written $p(z_s | z_{s^-}) = p(x_s, y_s | x_{s^-}, y_{s^-}) =$ $= p(x_s | x_{s^-}, y_{s^-}) p(y_s | x_s, x_{s^-}, y_{s^-})$, we see that HMT-IN is a PMT such that $p(x_s | x_{s^-}, y_{s^-}) = p(x_s | x_{s^-})$ and $p(y_s | x_s, x_{s^-}, y_{s^-}) = p(y_s | x_s)$.

3. DISCRETE HIDDEN PROCESS

Let us assume that $\Omega = \{\omega_1, ..., \omega_k\}$, which mean that the hidden process is a finite valued one. Let Z be a PMT defined with (2). Then the distribution p^y of X conditional to $Y = y$ keeps the same form (1). More precisely, for s child of s^-, we have [7] :

$$p^y(x_s | x_{s^-}) = \frac{\beta(x_s) p(z_s | z_{s^-})}{\sum_{\omega_s \in X} \beta(\omega_s) p(\omega_s, y_s | z_{s^-})} \qquad (3)$$

with the probabilities "backward" $\beta(x_s) = p(y_{s^{++}} | z_s)$ recursively calculable by

$$\beta(x_s) = 1 \text{ for } s^+ = \emptyset,$$
$$\beta(x_s) = \prod_{u \in s^+} (\sum_{\omega_u \in \Omega} \beta(\omega_u) p(\omega_u, y_u | z_s)) \text{ for } s^+ \neq \emptyset \qquad (4)$$

Otherwise, we have the following result, showing the greater generality of PMT with respect to HMT [7] :

Proposition 1

Let Z be a PMT defined with (2) and let (P) be the following property :

For each $s \in S - S^1, x_{s^-}, x_s \in \Omega$, and $y_{s^-} \in R$,

$$p(x_s | x_{s^-}, y_{s^-}) = p(x_s | x_{s^-}) \qquad \text{(P)}$$

Then :

1. (P) implies that Z is a HMT (i.e., X is a Markov tree);
2. Assume that each $s \in S$ has at least two children and for each $t_1 \in s^+$ there exists $t_2 \in s^+$ such that $p(z_{t_1} | z_s) = p(z_{t_2} | z_s)$ (the distributions of Z_{t_1} and Z_{t_2} conditional on Z_s are equal). Then " Z is a HMT" implies (P).

In particular, (P) is a necessary and sufficient condition when $p(z_s | z_{s^-})$ does not depend on $s \in (s^-)^+$.

An analogous result shows the greater generality of TMT with respect to PMT [8].

Let Z be a PMT defined with (2) and let us consider the problem of calculating the distribution of X_s conditional on $Y = y$ (marginal « a posteriori » distribution), needed when using the Bayesian Maximum a Posteriori (MPM) segmentation. This distribution $p(x_s | y)$ can be calculated in the following way. Let $s \in S$, and let $s_1 = s_r$, …, $s_n = s$ be the unique path (for each $2 \le i \le n$, s_{i-1} is the unique parent of s_i) leading from the root s_r to s. All $p^y(x_{s_i} | x_{s_{i-1}})$ having been calculated with (3), we have

$$p^y(x_s) = \frac{\alpha^s(x_s) \beta^s(x_s)}{\sum_{\omega_s \in \Omega} \alpha^s(\omega_s) \beta^s(\omega_s)} \qquad (5)$$

where $\alpha^s(x_s)$ is calculated using the path $s_1 = s_r$, …, $s_n = s$ by

$$\alpha^{s_1}(x_s) = \beta^{s_1}(x_s),$$
$$\alpha^{s_i}(x_{s_i}) = \sum_{\omega_{s_{i-1}} \in \Omega} p^y(x_{s_i} | x_{s_{i-1}}) \alpha^{s_{i-1}}(\omega_{s_{i-1}}) \qquad (6)$$

So, after having calculated $p(x_s | y)$ for each $s \in S$, one can use the classical Bayesian MPM segmentation method in which $\hat{x} = (\hat{x}_s)_{s \in S}$ is obtained by $\hat{x}_s = \arg\max_{\omega \in \Omega} p(x_s = \omega | y)$. When the segmentation is performed in an unsupervised manner, which is important in real applications, one has to estimate the model parameters from $Y = y$. The general methods like Expectaion-Maximization (EM [4]) or Iterative Conditional Estimation (ICE [10]) have been applied in the HMT and can be extended to the PMT and TMT cases. Classical HMT prove useful in statistical unsupervised segmentation problems [4]. The aim of the example presented in Figure 1 is to show that the greater generality of PMT can improve the results obtained with HMT. The class image is a 128x128 image and the Markov tree structure is a quad-tree [4]. So, we have the root and seven "generations", with the last generation S^7 being the set of 128x128 pixels. The noisy image $y^7 = (y_s)_{s \in S^7}$ is obtained by simulating a classical Gaussian noise on the generation S^6, and then using (2) to obtain $(y_s)_{s \in S^7}$. In the classical HMT case, we consider that only the last generation S^7 is noisy according to (1).

The results prented in Figure 1 are obtained in an unsupervised manner, the parameters being estimated with ICE.

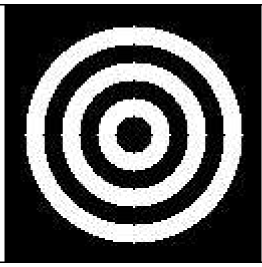
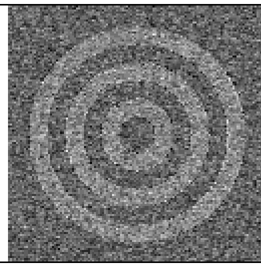
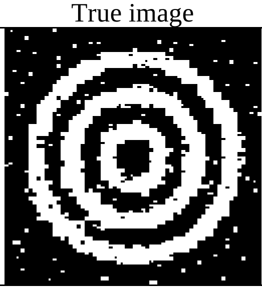
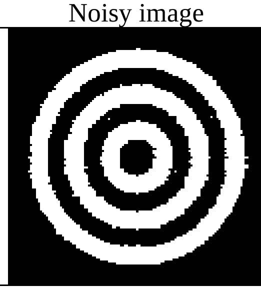
	
True image	Noisy image
	
HMT based method : error rate of 7.71%	PMT based method : error rate of 1.52%

Figure 1. HMT and PMT based unsupervised Bayesian segmentation results.

4. CONTINUOUS HIDDEN PROCESS

Let us now consider a PMT $Z=(X,Y)$, in which each X_s takes its values in R^N and each Y_s takes its values in R^q. Equations (3)-(6) still hold (with the obvious difference that sums should be replaced by intergrals), but may be difficult to compute in the general case.

So, let us now address the particular case in which in addition Z is a Gaussian process. Injecting this assumption in the algorithm of section 3 immediately leads to a Kalman-like smoothing algorithm which is omitted here for want of space.

In this section, we will rather show that in Gaussian case, it is also possible to develop a Kalman-like adaptive filtering algorithm for PMT. Recalling that S^1, .., S^n is a partition of S representing different « generations », let us put $X_n=(X_s)_{s\in S^n}$ and $X_{1:n}=(X_1,...,X_n)$. The random vectors Y_n, $Y_{1:n}$, Z_n, and $Z_{1:n}$ are defined similarly. Since Z is a Markov Tree, it can also be seen as a Markov Chain $Z=(Z_n)_{n\geq 1}$ to which the classical Kalman filter can thus be applied. More precisely, our aim consists in recursively estimating (as new data become available) the p.d.f. of the last "leaves" X_{n+1} given all observed variables up to level $n+1$, i.e. we want to compute $p(x_{n+1}|y_{n:n+1})$ in terms of $p(x_n|y_{n:n})$ and of y_{n+1}.

Our assumptions are as follows. We assume that the model is linear and Gaussian

$$\underbrace{\begin{bmatrix}X_s\\ Y_s\end{bmatrix}}_{Z_s}=\underbrace{\begin{bmatrix}F_s^1 & F_s^2\\ H_s^1 & H_s^2\end{bmatrix}}_{F_s}\underbrace{\begin{bmatrix}X_{s^-}\\ Y_{s^-}\end{bmatrix}}_{Z_{s^-}}+\underbrace{\begin{bmatrix}G_s^{11} & G_s^{12}\\ G_s^{21} & G_s^{22}\end{bmatrix}}_{G_s}\underbrace{\begin{bmatrix}U_s\\ V_s\end{bmatrix}}_{W_s} \qquad (7)$$

in which $W=(W_s)_{s\in S-S^1}$ are random zero-mean vectors, independent and independent of Z_1. We also assume that W is Gaussian and that Z_1 is Gaussian with mean $\bar{z}_1$ and variance-covariance matrix P_1, which is denoted by $Z_1 \sim \mathrm{N}(\bar{z}_1, P_1)$. Then Z is a Gaussian process and consequently the p.d.f. $p(x_{n+1}|y_{n:n+1})$ and $p(x_n|y_{n:n})$ are also Gaussian. For $l=0,1$, let us set $p(x_{n+l}|y_{n:n+1}) \sim \mathrm{N}(\hat{x}_{n+l|n}, P_{n+l|n})$ and let

$$E(W_sW_s^T)=Q_s,\ \tilde{Q}_s=\begin{bmatrix}\tilde{Q}_s^{11} & \tilde{Q}_s^{12}\\ \tilde{Q}_s^{21} & \tilde{Q}_s^{22}\end{bmatrix}=G_sQ_sG_s^T \qquad (8)$$

We shall also need the following notation : For n fixed, let $S^n=(s_1,...,s_k)$, and let $s_i^+=\{s_{i,p}^+\}_{p=1}^j$ (i.e. $s_{i,p}^+$ is the p^{th} son of node s_i). For $l,m\in\{1,2\}$, let F_{n+1}^l, H_{n+1}^l, and $\tilde{\mathrm{Q}}_{n+1}^{l,m}$ be the following block-diagonal matrices :

$$\begin{aligned}\mathrm{F}_{n+1}^l &= \mathrm{diag}(F_{s_1^+}^l,...,F_{s_k^+}^l),\\ \mathrm{H}_{n+1}^l &= \mathrm{diag}(H_{s_1^+}^l,...,H_{s_k^+}^l),\\ \tilde{\mathrm{Q}}_{n+1}^{lm} &= \mathrm{diag}(\tilde{Q}_{s_1^+}^{lm},...,\tilde{Q}_{s_k^+}^{lm}),\end{aligned} \qquad (9)$$

in which

$$F_{s_i^+}^l=\begin{bmatrix}F_{s_{i,1}^+}^l\\ \dots\\ F_{s_{i,j}^+}^l\end{bmatrix},\ H_{s_i^+}^l=\begin{bmatrix}H_{s_{i,1}^+}^l\\ \dots\\ H_{s_{i,j}^+}^l\end{bmatrix},\ \tilde{Q}_{s_i^+}^{lm}=\begin{bmatrix}\tilde{Q}_{s_i^+,1}^{lm} & & 0\\ & \dots & \\ 0 & & \tilde{Q}_{s_i^+,j}^{lm}\end{bmatrix} \qquad (10)$$

The following result is an extension of the classical Kalman filter

Proposition 2 (Kalman filter for PMT)

Let us assume that Z is a PMT and that model (7) holds. Suppose that $Z_1 \sim \mathrm{N}(\bar{z}_1, P_1)$ and $W_s \sim \mathrm{N}(0, Q_s)$.

Then $\hat{x}_{n+1|n+1}$ and $P_{n+1|n+1}$ can be computed from $\hat{x}_{n|n}$ and $P_{n|n}$ via :

Time-update equations

$$\hat{x}_{n+1|n} = F^1_{n+1}\hat{x}_{n|n} + F^2_{n+1} y_n , \quad (11)$$

$$P_{n+1|n} = \tilde{Q}^{11}_{n+1} + F^1_{n+1} P_{n|n} (F^1_{n+1})^T \quad (12)$$

Measurement-update equations

$$\tilde{y}_{n+1} = y_{n+1} - H^1_{n+1}\hat{x}_{n|n} - H^2_{n+1} y_n \quad (13)$$

$$L_{n+1} = \tilde{Q}^{22}_{n+1} + H^1_{n+1} P_{n|n} (H^1_{n+1})^T \quad (14)$$

$$K_{n+1|n+1} = (\tilde{Q}^{12}_{n+1} + F^1_{n+1} P_{n|n} (H^1_{n+1})^T) L^{-1}_{n+1} \quad (15)$$

$$\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} + K_{n+1|n+1}\tilde{y}_n \quad (16)$$

$$P_{n+1|n+1} = P_{n+1|n} - K_{n+1|n+1} L_{n+1} K^T_{n+1|n+1} \quad (17)$$

Remarks

- The algorithm is valid under the implicit assumption that each node has at least one child, but can easily by adapted to the general case where some node(s) have no child;
- If each root has exactly one child, then PMT reduces to a particular case of Pairwise Markov Chain model introduced in [5] (corollary 1, page 72), and the algorithm of the Proposition 2 reduces to the algorithm proposed for the latter model ([5], eqs. (13.56) and (13.57));
- The algorithm in Proposition 2 requires the inversion of the square matrix L_{n+1} defined in (13), the dimension of which is proportional to the number of variables in generation $n+1$ of the tree. However, this full-size matrix inversion can be avoided by conditioning w.r.t. each variable in y_{n+1} one after the other;
- In more general PMT which are neither linear nor Gaussian, one could consider to propose "Particle filtering", which would extend this king of methods proposed in the case of PMC [2] and TMC [3].

5. CONCLUSIONS

Recent PMT, strictly more general that HMT, can be used in discrete or continuous hidden signal restoration, as well in a supervised manner than in an unsupervised one. Its greater generality can lead to an improvement of the results obtained with the classical HMT. As further research we may mention the possibilities of extending PMT to Pairwise Markov Graphical models, with the associated methods of hidden process restoration and parameter estimation [14].

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