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Fast Image Segmentation with Contextual Scan and Markov Chains

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***Abstract*—Transforming bi-dimensional images into mono-dimensional sequences with Peano scan (PS) allows using Hidden Markov Chains (HMCs) for unsupervised image segmentation. In some situations, such methods can be competitive compared to Hidden Markov Fields (HMFs) based ones, while being much faster. We propose enriching the HMC-PS model by introducing “contextual” Peano scan (CPS). It consists in associating to each index in the HMC obtained from PS, two observations on pixels which are neighbors of the pixel considered in the image, but are not its neighbors in the HMC. This gives three observations on each point of the Peano scan, which leads to a new HMC with a more complex structure, but whose *prior* and *posterior* laws are still Markovian. Therefore we can apply the usual parameter estimation method: Stochastic Expectation-Maximization (SEM), as well as study unsupervised segmentation Marginal Posterior Mode (MPM) so obtained. The CPS based supervised and unsupervised MPM are compared to the classic scan based HMC-PS and the HMF through experiments on artificial images. They improve notably the former, and can even compete with the latter.**

Keywords—hidden Markov chains, Peano scan, unsupervised image segmentation, contextual Peano scan, SEM.

I. INTRODUCTION

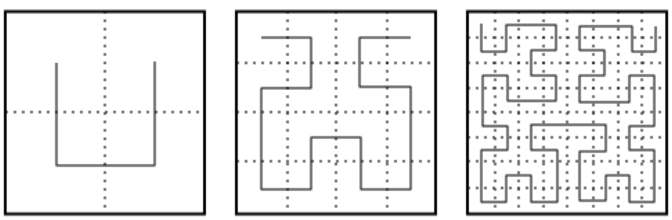
Unsupervised statistical image segmentation based on hidden Markov fields (HMFs) models are widely used since the pioneering papers [4, 13, 18]. HMFs are well adapted and give satisfactory results in numerous and various applications [9, 24, 25], among others. However, direct fast calculations are not tractable and one has to use iterative methods like Gibbs or Metropolis sampling. It significantly increases the computation time and makes the related methods improper in some situations. Using hidden Markov chains (HMCs [2, 7]) instead of HMFs is possible, however transforming bi-dimensional set of pixels into mono-dimensional sequence is not straightforward. For example, proceeding “line by line” gives a HMC such that pixels which are close in the set of pixels may be far in the sequence. Using Peano scan can partially overcome these difficulties and allows bringing closer the quality of the results obtained with the chains to the one obtained with the fields. Some comparison studies presented in [3, 11, 23] even show that in some situations HMCs based unsupervised segmentation can be competitive with HMFs based ones. The combination of Peano scan and HMCs has been used to segment different types of images. Let us mention radar ones [1, 11, 14], optical images [1, 22], or still MRI images [5, 6], where three-dimensional Peano scans are used. Peano scans have also been used in more sophisticated models than HMCs, like pairwise Markov chains with copulas [1], fuzzy Markov chains [8, 10, 15], HMCs with unknown number of states [20], multiresolution Markov chains [12], second order Markov chains [17], or still hidden semi-Markov chains [16]. In spite of the fact that data obtained from images via Peano scan have complex structure and are obviously not Markovian, the different mentioned methods can give quite interesting results, showing again the extraordinary robustness of HMCs and their extensions. All studies mentioned above show the interest of the Peano scan in problems where the computer time is of importance; indeed, thanks to direct, recursive and exact computations, HMCs based unsupervised segmentations are incomparably faster than HMFs based ones.

In this paper, we aim to improve the efficiency of the HMCs based methods by extending the Peano scan based model according to the following idea. Let s be a pixel in image, and let r, t, u, w be its four nearest neighbors. Let r, t be its neighbors in Peano scan. Then we propose a model in which the observations on remaining neighbors u, w are also taken into account in the Peano scan: the image value observed on s is completed by the two observations on u and w. We show that using HMCs in such a “contextual” Peano scan framework allows reducing the classification error by up to seventeen percent. In addition, some experiments show the existence of situations in which the new model is competing with the classic HMF model at the efficiency level, while being much faster. Applying a stochastic version of the expectation-maximisation (EM) algorithm [19] to the new model, we notice its efficiency in some simple synthetic image segmentation cases studied. We show examples in which the new model based MPM takes the upper hand over well-known HMF model.

The organization of the paper is as follows. In the next section we present the contextual Peano scan and related HMC. We recall the classic Bayesian Maximum Posterior Mode (MPM) segmentation in section three, and parameter estimation with the Stochastic Expectation-Maximisation (SEM) algorithm in specified in section four. Fifth section is devoted to experiments, and last section contains conclusions and perspectives.

II. Contextual Peano Scan and related Stochastic Chain

Let $X^N = (X_1, \dots, X_N)$ be the sequence obtained by the Peano scan, whose construction is presented on Fig. 1. For each $n = 2, \dots, N-1$ let us set s the corresponding pixel in S. The past point $n-1$ will be called r, and the next point $n+1$ will be called t. Then we associate to each $n = 2, \dots, N-1$ one u_n and one w_n, which are two neighbors of n different from r and t. Thus each $n = s$ has four neighbors in S: two $r = n-1$ and $t = n+1$ which belong to the Peano scan, and two u_n and w_n, which don't. Then for each X_n we associate the triplet $Y_n^* = (Y_{u_n}, Y_n, Y_{w_n})$.



(a) (b) (c)

Fig. 1. Construction of Peano scan.

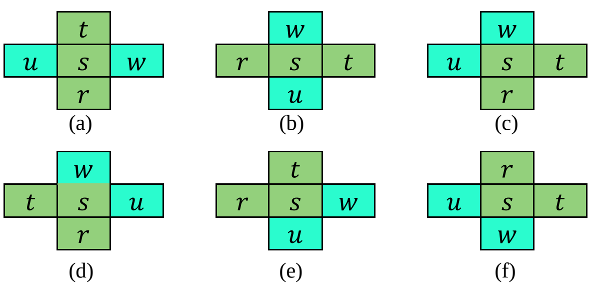


Fig. 2. Six spatial configurations of added neighbors. In green: neighbors of s being neighbors in Peano scan; in blue: neighbors of s in the set of pixels whose observations are associated with observation y_s on s, and which are not neighbors of s in Peano scan.

Example 1

As an example, let us consider image (b) in Fig. 1, with the Peano scan beginning in the upper left corner. Points $n = 1, \dots, 16$ are specified in Fig. 3, and added observations are specified in Fig. 4.

1	2	15	16
4	3	14	13
5	8	9	12
6	7	10	11

Fig. 3. Pixels numbering in image (b) in Fig. 1, with the Peano scan beginning in the up left corner.

1	2	3	4	5	6	7	8
y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
-	-	y_{14}	-	-	-	-	y_3
y_4	y_{15}	y_8	y_1	y_8	-	y_{10}	y

9	10	11	12	13	14	15	16
y_9	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	y_{16}
y_{12}	-	-	-	-	y_9	y_2	y_{13}
y_{14}	y_7	-	y_9	y_{16}	y_3	-	-

Fig. 4. Observations associated with pixels 1, 2, 3, …., 16 in Fig. 3, which are (y_1, y_4), (y_2, y_3), (y_{14}, y_3, y_8), (y_1, y_4), (y_5, y_8), y_6, (y_7, y_{10}), (y_8, y_3, y_5), …, (y_{16}, y_{13}),

Let s, t be neighbors in S. If they are horizontal neighbors, let us set

$$p^h(y_t|x_s) = \sum_{x_t} p^h(x_t|x_s) p(y_t|x_t) \quad (1)$$

Similarly, if they are vertical neighbors, we set

$$p^v(y_t|x_s) = \sum_{x_t} p^v(x_t|x_s) p(y_t|x_t) \quad (2)$$

Thus for s in S and u, w neighbors of s in the set of pixels, but not neighbors in the Peano scan, we have

$$p(y_u, y_s, y_w|x_s) = p(y_s|x_s) p^{a(u,s)}(y_u|x_s) p^{a(w,s)}(y_w|x_s), \quad (3)$$

where $a(u, s) = h$ if u, s are horizontal neighbors, and $a(u, s) = v$ if u, s are vertical neighbors, and the same for $a(w, s)$.

For example, considering (y_{14}, y_3, y_8) in Fig. 4 we see that (y_{14}, y_3) are horizontal neighbors, while (y_3, y_8) are vertical neighbors. Then we have

$$p(y_{14}, y_3, y_8|x_3) = p(y_3|x_3) p^h(y_{14}|x_3) p^v(y_8|x_3) \quad (4)$$

Finally, for a given Peano scan, we associate with each pixel s the two neighbors $u(s)$, $w(s)$ in the set of pixels, which are not its neighbors in the Peano scan. Numbering the Peano scan points as $(1, 2, \dots, N)$, the related contextual Peano scan is the sequence of triplets

$$([1, u(1), w(1)], [2, u(2), w(2)], \dots, [N, u(N), w(N)]), \quad (5)$$

The classic hidden Markov chain associated with the Peano scan has a distribution:

$$p(x^N, y^N) = p(x_1) p(x_2|x_1) \dots p(x_N|x_{N-1})\, p(y_1|x_1) \dots p(y_N|x_N) \quad (6)$$

The new model we propose, called "conditional Markov chain for contextual Peano scan" (CMC-CPS), is defined as follows. Let us consider:

$$q(x^N, y^N) = p(x_1) \prod_{n=1}^{N-1} p(x_{n+1}|x_n) \prod_{n=1}^{N} p\big(y_n, y_{u(n)}, y_{w(n)}\big|x_n\big) \quad (7)$$

It is to be noted that $q(x^N, y^N)$ is not the probability density of the pair (X^N, Y^N). Nonetheless, considering the (unknown) normalizing constant:

$$\kappa = \frac{1}{\sum_{x^N} \int q(x^N, y^N) dy^N}$$

we can define the law of the CMC-CPS with:

$$p(x^N, y^N) = \kappa q(x^N, y^N) \quad (8)$$

Definition

Let S be a square set of pixels of dimensions $N = 2^k \times 2^k$. Let $(1, 2, \dots, N)$ be a Peano scan (PS) of S, and let $([1, u(1), w(1)], [2, u(2), w(2)], \dots, [N, u(N), w(N)])$ be the four nearest neighbors contextual PS (4NN-CPS) associated with PS.
Then the conditional probability distribution $p(x^N|y^N)$ given from the distribution $p(x^N, y^N) = \kappa q(x^N, y^N)$ defined with (3), (7), and (8) will be called "conditional Markov chain for contextual Peano scan" (CMC-CPS) distribution.

Remark 1

The name CMC-CPS specifies that the hidden chain X^N is Markovian conditionally on Y^N. As we will see in the next section $p(x_1|y^N)$ and transitions $p(x_{n+1}|x_n, y^N)$ are computable, which will allow Bayesian restorations.

Remark 2

It is possible to extend the 4NN-CPS with richer neighborhood. Considering eight nearest neighbors in the set of pixels would lead to a contextual PS with six additional observations on each point in Peano scan.

III. BAYESIAN MAXIMUM POSTERIORI MODE SEGMENTATION

Let $1, \dots, N$ be points of a Peano scan, and let $X^N = (X_1, \dots, X_N)$ be a Markov chain. The hidden image of classes we look for is then considered as being a realization of X^N. According to the construction in the previous section, let us consider $Y^N = (Y_1, \dots, Y_N)$ with Y_n denoting the value of observed image on pixel n. The pairwise stochastic process $(X^N, Y^N) = (X_1, Y_1, \dots, X_N, Y_N)$ so obtained is not a classic HMC and $p(y^N|x^N)$ has complex structure (see Remark 3 below); however, as $p(x^N|y^N)$ is Markovian, Bayesian segmentation is still workable.

The Bayesian Marginal Posterior Mode (MPM) we consider is defined by

$$[\hat{s}_{MPM}(y^N) = \hat{x}^N] \Leftrightarrow \left[p(\hat{x}_n|y^N) = \max_{x_n \in \Omega} p(x_n|y^N)\right]; \quad (9)$$

thus the problem lies in computing $p(x_n|y^N)$.

Let us recall the following general result.

Proposition

Let $Z^N = (Z_1, \dots, Z_N)$ be a stochastic chain taking its values in a finite set Ω. Then:

(i) Z^N is Markov if and only if there exist $N-1$ functions φ_1, …, φ_{N-1} from Ω^2 to R^+ such that

$$p(z^N) \propto \prod_{n=1}^{N-1} \varphi_n(z_n, z_{n+1}), \quad (10)$$

with $\propto$ meaning "proportional to";
(ii) for (10) verified, $p(z_1)$ and transitions $p(z_{n+1}|z_n)$ are given from functions φ_1, …, φ_{N-1} with

$$p(z_1) = \frac{\beta_1(z_1)}{\sum_{z_1} \beta_1(z_1)}; \quad (11)$$

$$\text{for } 1 < n < N,\ p(z_{n+1}|z_n) = \frac{\varphi_n(z_n, z_{n+1})\beta_{n+1}(z_{n+1})}{\beta_n(z_n)},$$

where $\beta_n(z_n)$ can be computed with the backward recursion:

$$\beta_N(z_N) = 1\,; \text{for} = N, \dots, 2\,,$$

$$\beta_{n-1}(z_{n-1}) = \sum_{z_n} \varphi_n(z_{n-1}, z_n)\beta_n(z_n), \quad (12)$$

Once $p(z_1)$ and $p(z_{n+1}|z_n)$given, each $p(z_n)$ is computed with forward recursion:

$$\text{for} = 2, \dots, N\,,\ p(z_n) = \sum_{z_n} p(z_n|z_{n-1})p(z_{n-1}). \quad (13)$$

To summarize, once functions φ_1, …, φ_{N-1} verifying (10) are given, marginal distributions $p(z_n)$ of the Markov chain $Z^N = (Z_1, \dots, Z_N)$ are computable.
Let us return to the conditional Markov chain for contextual Peano scan distribution $p(x^N|y^N)$ specified in Definition. We can say that $p(x^N|y^N) \propto q(x^N, y^N)$, and thus

$$p(x^N|y^N) \propto \prod_{n=1}^{N-1} \varphi_n(x_n, x_{n+1}, y^N), \quad (14)$$

with

$$\varphi_1(x_1, x_2, y^N) = p(x_1, x_2)p(y_1, y_{u(1)}, y_{w(1)}|x_1)p(y_2, y_{u(2)}, y_{w(2)}|x_2);$$
$$\varphi_2(x_2, x_3, y^N) = p(x_3|x_2)p(y_3, y_{u(3)}, y_{w(3)}|x_3); \quad (15)$$
$$\dots.$$
$$\varphi_{N-1}(x_{N-1}, x_N, y^N) = p(x_N|x_{N-1})p(y_N, y_{u(N)}, y_{w(N)}|x_N).$$

Finally, functions φ_1, …, φ_{N-1} verifying (2) are of the form

$$\varphi_1(x_1, x_2, y^N) = \varphi_1(x_1, x_2, y_1, y_{u(1)}, y_{w(1)}, y_2, y_{u(2)}, y_{w(2)});$$
$$\varphi_2(x_2, x_3, y^N) = \varphi_2(x_2, x_3, y_3, y_{u(3)}, y_{w(3)});$$
$$\dots$$
$$\varphi_{N-1}(x_{N-1}, x_N, y^N) = \varphi_{N-1}(x_{N-1}, x_N, y_N, y_{u(N)}, y_{w(N)}).$$

and thus are easy to compute. Then $p(x_1|y^N)$ and transitions $p(x_{n+1}|x_n, y^N)$ are computable, which gives marginal distributions $p(x_n|y^N)$ of $p(x^N|y^N)$ and allows the use of MPM (9).

Remark 3

The pair (X^N, Y^N) has complex and only partially known structure. $p(x^N)$ is Markovian, but neither $p(x^N, y^N)$ nor $p(y^N|x^N)$ is. In addition, for $p(y_n|x_n)$ Gaussian, the distribution $p(y^N|x^N)$ is not Gaussian in general. However all this matter little as the important thing is that $p(x^N|y^N)$ is Markovian with computable $p(x_n|y^N)$.

Example 2

Let us specify $\varphi_1, \varphi_2, \ldots, \varphi_{15}$ used in Example 1. According to Fig. 3, and (3), (15) we have:

$\varphi_1(x_1, x_2, y^N) =$
$p(x_1, x_2)p(y_1|x_1)p^v(y_4|x_1)p(y_2|x_2)p^h(y_{15}|x_2)$;

$\varphi_2(x_2, x_3, y^N) = p(x_3|x_2)p(y_3|x_3)p^v(y_8|x_3)p^h(y_{14}|x_3)$;

$\varphi_3(x_3, x_4, y^N) = p(x_4|x_3)p(y_4|x_4)p^v(y_1|x_4)$;

…

$\varphi_{13}(x_{13}, x_{14}, y^N) =$
$p(x_{14}|x_{13})p(y_{14}|x_{14})p^v(y_9|x_{14})p^h(y_3|x_{14})$;

$\varphi_{14}(x_{14}, x_{15}, y^N) = p(x_{15}|x_{14})p(y_{15}|x_{15})p^h(y_2|x_{15})$;

$\varphi_{15}(x_{15}, x_{16}, y^N) = p(x_{16}|x_{15})p(y_{16}|x_{16})p^v(y_{13}|x_{16})$.

IV. Parameter Estimation

Let us suppose that $p(x^N, y^N)$ is a classic hidden Markov chain (CHMC) distribution, with Gaussian $p(y^N|x^N)$ and two different transitions depending on whether it applies to horizontal neighbors or vertical neighbors in the original image. For K classes $\Omega = \{1, \ldots, K\}$ the parameters are: K^2 probabilities $p^h = (p_{ij}^h)_{1\le i,j\le K}$, with $p_{ij}^h = p(x_{t+1} = i, x_t = j)$ for i, j neighbors in the chain and horizontal neighbors in the image, K^2 probabilities $p^v = (p_{ij}^v)_{1\le i,j\le K}$, with $p_{ij}^v = p(x_{t+1} = i, x_t = j)$ for i, j neighbors in the chain and vertical neighbors in the image, K means $m = (m_i)_{1\le i\le K}$ and K variances $\sigma^2 = (\sigma_i^2)_{1\le i\le K}$ of the K Gaussian distributions $(p(y_s|x_1 = i))_{1\le i\le K}$. By choosing $p^h(x_t|x_s)$ and $p^v(x_t|x_s)$ intervening in $p(y_u, y_s, y_w|x_s)$ to be equal to p_{ij}^h and p_{ij}^v respectively, our new model use exactly the same parameters than the CHMC, so the problem is to estimate $\theta = (p^h, p^v, m, \sigma^2)$ from the observed image $Y^N = y^N$ alone. As $p(x^N|y^N, \theta^q)$ is computable and it is possible to sample from it, we can use Stochastic EM (SEM), which runs as follows.

1. Initialize the parameters $\theta^0 = (p^{h,0}, p^{v,0}, m^0, \sigma^{2,0})$ with some simple method;
2. Compute $\theta^{q+1} = (p^{h,q+1}, p^{v,q+1}, m^{q+1}, \sigma^{2,q+1})$ from current $\theta^q = (p^{h,q}, p^{v,q}, m^q, \sigma^{2,q})$ and $y^1, y^2, \ldots, y^L$:

- Sample $x^{N,q+1} = (x_1^{q+1}, \ldots, x_N^{q+1})$ according to the Markov distribution $p(x^N|y^N, \theta^q)$;
- Let H^{q+1} be the set of couples $(n, n+1)$, with n and $n+1$ horizontal neighbors in the set of pixels, and let V^{q+1} be the set of couples $(n, n+1)$, with n and $n+1$ vertical neighbors in the set of pixels. Let $S^{i,q+1}$ be the set of points n such that $x_n^{q+1} = i$. We have

$$p_{ij}^{h,q+1} = \frac{\sum_{(n,n+1)\in H^{q+1}} 1_{[x_n^{q+1}=i, x_{n+1}^{q+1}=j]}}{|H^{q+1}|}, \text{ similar for } p_{ij}^{v,q+1}, \quad (16)$$

$$m_i^{q+1} = \frac{\sum_{n\in S^{i,q+1}} y_n}{|S^{i,q+1}|}, \quad \sigma_{i,q+1}^2 = \frac{\sum_{n\in S^{i,q+1}} (y_n - m_i^{q+1})^2}{|S^{i,q+1}|}. \quad (17)$$

Let us notice that one can use a simpler approximated SEM, by treating $p(x^N, y^N)$ as a CHMC, and sampling from its posterior law $p^{\text{CHMC}}(x^N|y^N, \theta^q)$ instead of the real $p(x^N|y^N, \theta^q)$; we programmed it and it gives slightly worst results.

V. Experiments

We propose a segmentation study aiming to answer the following questions: (i) does the new CPS model based on contextual Peano scan work better than the classic Peano scan based one? (ii) how the new model works with respect to hidden Markov field model? (iii) is SEM efficient? We present segmentation results and error ratios on Fig. 5.

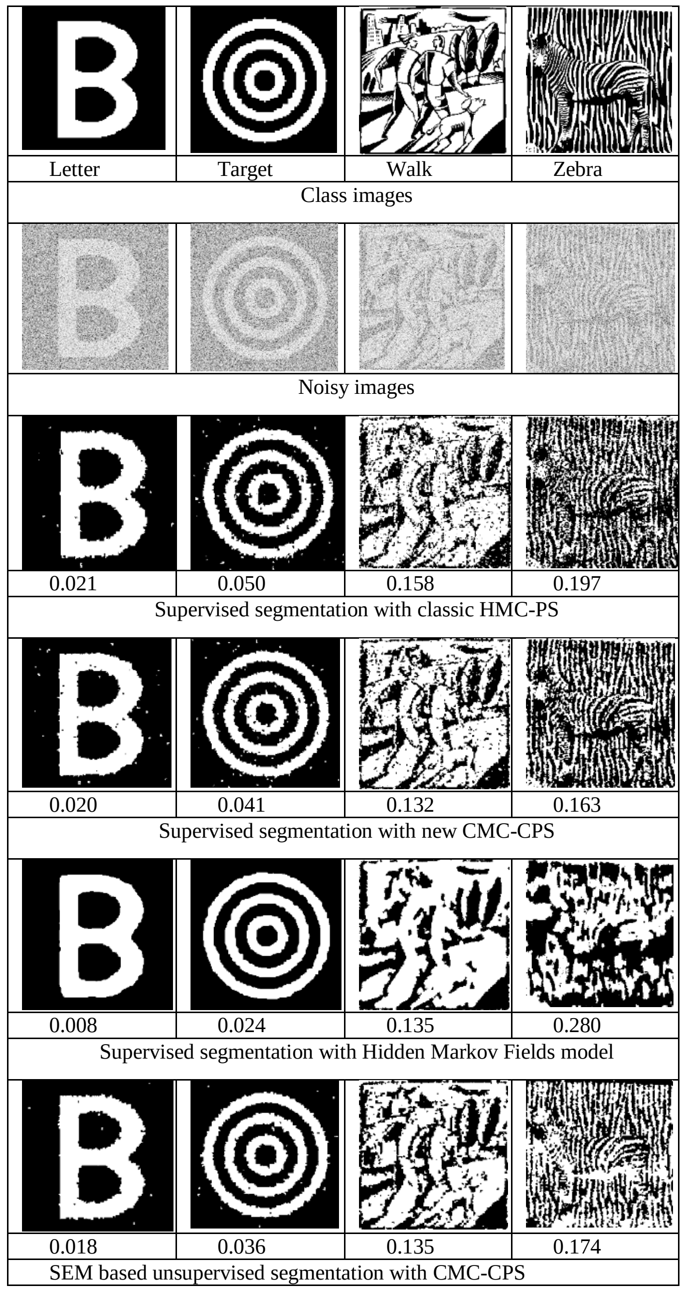


Fig. 5. Segmentation of four class images noised with two Gaussian noise distributions with means 0, 1, and common variance 1. Numbers are error ratios.

We call "supervised" the MPM using real noise parameters and parameters of $p(x^N)$ estimated from x^N, while "unsupervised" MPM uses parameters estimated from by SEM. Besides, we only consider the case $p_{ij}^h = p_{ij}^v$.

Concerning point (i) we notice that the new method based on contextual Peano scan and related new model (CMC-CPS) always improves results obtained with the classic method based on HMC and Peano scan (HMC-PS). The gain is modest, and may even be negligible, in the case of homogeneous (in the sense that the same class areas are of large size) image like Letter. However, it is significant in the three remaining cases, where the error ratio decreases by about 17%.

Point (ii) is of main importance. HMFs keep the upper hand for homogeneous images like Letter or Target; however, they struggle to process fine details in Walk and Zebra, and are downgraded by CMC-CPS. Of course, CMC-CPS is much faster: setting 100 samplings with Gibbs sampler, each sampling obtained with 100 scans, Markov field based MPM takes about two and half hours, while CMC-CPS, needing no iterations, takes about one second. Moreover, the Markov field considered directly uses the distributions conditional to the four nearest neighbors as proposed in [9], and thus it has five degrees of freedom for the parameters, while the Markov chain in CMC-CPS has only three degrees of freedom for the parameters. This shows that the complex structure of the noise in CMC-PS is well adapted to the problem. Besides, we notice that when using CMC-CPS, unsupervised MPM may give better results that supervised one. This is somewhat surprising, but possible.

Finally, we can answer to the last point (iii) noticing that ratio errors obtained with unsupervised CMC-CPS's based MPM are close to - or even better than - those obtained with supervised one.

VI. Conclusion and Perspectives

We presented a fast unsupervised segmentation Bayesian MPM method based on the new contextual Peano scan and related extension of hidden Markov chains. Compared to the classic Peano scan based methods the new one can reduce the segmentation error up to 20 %. In addition, the new method can give comparable results to those obtained with the hidden Markov fields based one, while being much faster.

Extending the model considered to triplet Markov chains [21] opens huge perspectives that we hope develop in the next years.

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