Existence of Stationary Points for Pseudo-Linear Regression Identification Algorithms

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Abstract—The authors prove existence of a stable transfer function satisfying the nonlinear equations characterizing an asymptotic stationary point, in undermodeled cases, for a class of pseudo-linear regression algorithms, including Landau’s algorithm, the Feintuch algorithm, and (S)HARF. The proof applies to all degrees of undermodeling and assumes only that the input power spectral density function is bounded and nonzero for all frequencies and that the compensation filter is strictly minimum phase. Some connections to previous stability analyses for reduced-order identification in this algorithm class are brought out.

Index Terms—Identification, pseudo-linear regression, undermodeled stationary points.

I. INTRODUCTION

Many convergence results for adaptive identification algorithms are limited to sufficient order settings, in which an unknown system is of finite and known order. Fewer results are available for more realistic undermodeled (or reduced-order) settings, in which the degree of an unknown system is underestimated or prohibitively large. Here we prove, for undermodeled cases, the existence of a stable transfer function satisfying the nonlinear equations governing a mean stationary point for pseudo-linear regression algorithms, including Landau’s identifier [1], the Feintuch algorithm [2], and (S)HARF [3], [4].

The importance of examining the behavior of adaptive identification algorithms in undermodeled cases has been recognized repeatedly over the years [5]–[8], since physical systems that one may attempt to identify need not admit exact descriptions using finite-order models. Two contributions by Anderson and Johnson [5], [6] show the absence of divergence for this algorithm class in undermodeled cases, subject to excitation and strict positive real conditions. Those results tacitly assume existence of a reduced-order model that the algorithm may be attempting to identify, without specifying its structure. As argued in [8], a natural choice for the reduced-order model is the system obtained at a mean stationary point, if one exists. Section II reviews this question and states our main result asserting the existence of a stationary point giving rise to a stable transfer function. Section III presents the technical construct, with proofs of intermediate lemmas deferred to Section IV. Concluding remarks are synthesized in Section V.

II. PROBLEM STRUCTURE AND MAIN RESULT

We consider a system identification problem in which \( \{u(\cdot)\} \) is a discrete-time zero mean stationary stochastic process, and the observed system output \( \{y(\cdot)\} \) is generated according to

\[
y(n) = \sum_{k=0}^{\infty} h_k u(n-k) + \zeta(n)
\]

where \( \{\zeta(\cdot)\} \) is a stationary output disturbance assumed independent of the input sequence \( \{u(\cdot)\} \), and \( \{h_k\} \) is the impulse response of the unknown system, assumed stable and causal. Its transfer function

\[
H(z) = \sum_{k=0}^{\infty} h_k z^k
\]

where \( z \) is the backward delay operator: \( z u(n) = u(n-1) \). This transfer function may be irrational.

The SHARF algorithm [3] adjusts the coefficients of a candidate rational model

\[
\hat{H}(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z + \cdots + b_M z^M}{1 + a_1 z + \cdots + a_M z^M}
\]

using the following adaptation algorithm:

\[
\begin{align*}
\hat{y}(n) &= -\sum_{k=1}^{M} a_k(n) \hat{y}(n-k) + \sum_{k=0}^{M} b_k(n) u(n-k) \\
\epsilon(n) &= \sum_{k=0}^{M} c_k [y(n-k) - \hat{y}(n-k)], \quad c_0 = 1 \\
a_{k+1}(n+1) &= a_k(n) - \mu \epsilon(n) \hat{y}(n-k), \quad k = 1, 2, \ldots, M \\
b_{k+1}(n+1) &= b_k(n) + \mu \epsilon(n) u(n-k), \quad k = 0, 1, \ldots, M.
\end{align*}
\]

Here \( \mu > 0 \) is a small adaptation step size, and \( C(z) = 1 + c_1 z + \cdots + c_M z^M \) is a compensation filter with user-chosen coefficients \( \{c_k\} \). Variants of the basic algorithm include HARF [4] using a posteriori filtered regressors, Landau’s algorithm [1] using a matrix gain sequence in place of \( \mu \), and the Feintuch algorithm in which \( C(z) = 1 \). This algorithm class is globally convergent (strongly [9] or weakly [10]) in the sufficient order case (i.e., when \( \deg H(z) \leq M \)), provided the compensation filter is chosen correctly.

Convergence behavior in more realistic undermodeled (or reduced-order) cases is more difficult to analyze, since the algorithm is not a gradient descent procedure, although results are available in [5] and [6]. With respect to Fig. 1, suppose the larger order system \( H(z) \) is split into a reduced-order part, of degree not exceeding \( M \) but otherwise unspecified plus the remaining undermodeled dynamics and measurement noise. If the undermodeled dynamics and measurement noise are momentarily set to zero, we recover a sufficient-order setting. Global asymptotic convergence may be shown based on passivity applied to an internal energy function. Once the undermodeled dynamics and measurement noise are added back to the system, the

![Fig. 1. Decomposition of unknown system.](image-url)
internal energy function is shown to remain asymptotically bounded under the excitation conditions specified in [5], [6], which implies the absence of divergence.

Let \( B_z(z) / A_z(z) \) be the (as yet unspecified) reduced-order subsystem in Fig. 1. The passivity arguments from [5] and [6] assume that the compensation filter \( C_z(z) \) is chosen such that \( C_z(z) / A_z(z) \) is strictly positive real (SPR). To the extent that the structure of the reduced-order part \( B_z(z) / A_z(z) \) is unspecified, so is the set of admissible compensation filters which ensure stability of the adaptive system.

As argued in [8], some meaning can be given to the decomposition of Fig. 1 provided the reduced-order system is that obtained at a mean asymptotic stationary point of the algorithm, if one exists. If so, then the energy function from [5] and [6] would measure the deviation between the parameter estimates and their mean values, and bounding the energy function would constrain the identifier to stay in a neighborhood of the stationary point. A mean asymptotic stationary point corresponds, of course, to those parameter values \( \{a_k\} \) and \( \{b_k\} \) which, if held fixed, would render the error \( \epsilon(n) \) orthogonal to the regressor signals, i.e.,

\[
E[\epsilon(n)u(n-k)] = 0, \quad k = 0, 1, \ldots, M \tag{1}
\]

\[
E[\epsilon(n)\tilde{g}(n-k)] = 0, \quad k = 1, 2, \ldots, M. \tag{2}
\]

These equations may be obtained by averaging to the classical theory (e.g., [11]) or stochastic approximation theory (e.g., [10]). Either way, these equations are nonlinear in the filter coefficients \( \{a_k\} \), and the existence of a stable transfer function \( \tilde{H}(z) \) consistent with (1) and (2) has not previously been established for the underdetermined case. Our contribution is summarized in the following theorem.

**Theorem 1:** Suppose the input power spectral density function

\[
S(z) = \sum_{k=-\infty}^{\infty} E[u(n)u(n-k)]z^k
\]

is nonzero and bounded for all \( |z| = 1 \), and that the compensation filter \( C_z(z) \) is strictly minimum phase (no zeros in \( |z| \leq 1 \)). Equations (1) and (2) then admit a solution for which the resulting \( \tilde{H}(z) \) is a stable transfer function.

**Remark I:** In many studies [1], [3]–[6], [9], \( C_z(z) \) generates the numerator of a transfer function which should be SPR for convergence to apply; this constrains \( C_z(z) \) to be strictly minimum phase. The proof of Theorem 1, however, invokes no SPR condition.

### III. The Existence Proof

We shall work in the transfer function space \( L_2 \), consisting of functions \( f(z) = \sum_{i=-\infty}^{\infty} f_i z^i \) having square-cumulative coefficients \( \{f_i\} \). The causal projection (onto the Hardy space \( H_2 \)) is denoted as \( [f(z)]^{+} = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \cdots \), yielding a function analytic in \( |z| < 1 \). If \( f(z) \) and \( g(z) \) are (column) vector-valued, the inner product

\[
(f(z), g(z)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\omega})^* g(e^{i\omega}) d\omega = \sum_{i=-\infty}^{\infty} f_i g_i^* \tag{3}
\]

in which \( \dagger \) denotes conjugate transposition, becomes a matrix containing all pairwise scalar products, with immediate specialization to the scalar case.

Introduce now the one-sided \( z \)-transform

\[
\sum_{n=0}^{\infty} E[\epsilon(n)u(n+i)]z^i = [S(z)C(z)]\tilde{H}(z) \tag{4}
\]

If the output noise \( \{\zeta(z)\} \) is independent of the input \( \{u(z)\} \), then it makes no contribution to this expression. The following result on certain zeros of this function is adapted from [8, Sec. 9.7].

**Theorem 2:** If the rational function \( \hat{H}(z) \) fulfills

\[
[S(z)C(z)\tilde{H}(z) - \hat{H}(z)]^{+} = z^{M+1} V_M(z)Q(z) \tag{5}
\]

for some stable and causal \( Q(z) \), where

\[
V_M(z) = \frac{a_M + a_{M-1} z + \cdots + a_1 z^{M-1} + z^{M}}{1 + a_1 z + \cdots + a_{M-1} z^{M-1} + a_M z^M}
\]

is a causal all-pass function whose zeros are the reciprocals of the poles of \( \tilde{H}(z) \), then (1) and (2) are satisfied.

We show that there always exists a stable rational function \( \hat{H}(z) \), of degree not exceeding \( M \), which fulfills this characterization. As this problem is nonlinear in the filter coefficients \( \{a_k\} \), we may reparameterize \( \tilde{H}(z) \) so that the set of stable and causal functions occupies a convex region in the parameter space, using a normalized lattice parameterization [12].

To this end, we set \( \lambda_3 h(z) = 1 + a_1 z + \cdots + a_{M-1} z^{M-1} + a_M z^M \) and \( \lambda_3 h(z) = z^M \lambda_3 h(z^{-1}) \) and then determine lower order polynomials from the Schur recursion

\[
\begin{bmatrix}
A_{k-1}(z) \\
A_k(z)
\end{bmatrix} = \begin{bmatrix}
1 & \cos \theta_k \\
-\sin \theta_k & 0
\end{bmatrix} \begin{bmatrix}
A_{k-1}(z) \\
A_k(z)
\end{bmatrix} \tag{6}
\]

in which \( \sin \theta_k = \lambda_3 h(0)/A_k(0) \) and \( \cos \theta_k \) is nonnegative. It is well known [15] that this procedure will iterate successfully \( M \) times, yielding \( \sin \theta_k < 1 \) for \( k = M, \ldots, 1 \), if and only if \( \lambda_3 h(z) \) is strictly minimum phase (all zeros in \( |z| > 1 \)). Let us set \( s_k = \sin \theta_k \) and \( s = [s_1, s_2, \ldots, s_M] \); the stability domain becomes the following convex open subset of \( \mathbb{R}^M \):

\[
D = \{s : |s_k| < 1 \text{ for } k = 1, 2, \ldots, M\} \tag{7}
\]

Let \( V_k(z) = \lambda_3 h(z) / A_{M}(z) \) for \( k = 0, 1, \ldots, M \), and \( V(z) = \left[ V_0(z), \ldots, V_M(z) \right]^T \). Using the Beurling–Lax theorem [13], [8], one may show that the orthogonal complement in \( H_2 \) to the space spanned by \( V_0(z), \ldots, V_M(z) \) consists of all shifted versions of \( z V_M(z) \), i.e.,

\[
\lambda_{M+1} = \left( V(z), f(z) \right) \Leftrightarrow [f(z)]^{+} = z^M V_M(z)g(z) \tag{8}
\]

for some \( g(z) \in H_2 \).

Given any numerator polynomial \( B(z) = b_0 + b_1 z + \cdots + b_M z^M \), we may always determine coefficients \( v_0, \ldots, v_M \) such that

\[
\hat{H}(z) = \frac{B(z)}{A_M(z)} = \sum_{k=0}^{M} v_k V_k(z)
\]

and then work with the parameters \( \{s_k\} \) and \( \{v_k\} \) rather than the coefficients \( \{a_k\} \) and \( \{b_k\} \).

We now state the chain of arguments which will establish Theorem 1. First we constrain the zeros of \( \tilde{H}(z) \) as a function of the poles, in order to obtain a partial fulfillment of the frequency domain characterization (3). For given values \( \{s_k\} \), which determine the denominator \( A_M(z) \), we may find coefficients \( \{v_k\} \)—dependent on \( \{s_k\} \)—for which

\[
[S(z)C(z)[H(z) - \hat{H}(z)]^{+} = z^M V_M(z)R(z) \tag{9}
\]

for some \( R(z) \in H_2 \). To verify, we note from (6) that this amounts to choosing the coefficients \( \{v_k\} \) such that

\[
\lambda_{M+1} = \left( V(z), S(z)C(z)[H(z) - \hat{H}(z)] \right) \tag{10}
\]

Expanding \( \hat{H}(z) = \sum_{k=0}^{M} v_k V_k(z) \), this equality reduces to the linear system

\[
\left( V(z), S(z)C(z)H(z) \right) = \left( V(z), S(z)C(z) \right) \begin{bmatrix}
v_0 \\
\vdots \\
v_M
\end{bmatrix} \tag{11}
\]
We show in the Appendix that the matrix on the right-hand side is invertible whenever \( C(z) \) is strictly minimum phase, which yields a unique solution for \( \{v_k\} \).

The zeros of \( \hat{H}(z) = \sum_{k=0}^M v_k V_k(z) \) are now constrained with respect to the poles, which are set by the Schur parameters \( s = \{s_k\} \). We write \( V_k(z) \) and \( \hat{H}(z, s) \) to emphasize this dependence.

To complete our construction of \( \hat{H}(z) \) fulfilling (3), we must now choose the Schur parameters \( s \) such that the remainder function \( R(z) \) from (7) vanishes in its first \( M \) coefficients, i.e., \( R(z) = z^M Q(z) \) for some \( Q(z) \in \mathbb{R}[z] \). This in turn can be written as

\[
0 = (z^k, R(z)), \quad k = 0, 1, \ldots, M - 1.
\]  

But from (7) we can write \( R(z) \) as

\[
R(z) = z^{-1} V_{M_z}(z^{-1}, s)[S(z)C(z)[\hat{H}(z) - \hat{H}(z, s)]]^+
\]

so that (9) becomes

\[
0 = (z^k, z^{-1} V_{M_z}(z^{-1}, s)[S(z)C(z)[\hat{H}(z) - \hat{H}(z, s)]]^+),
\]

for \( k = 0, 1, \ldots, M - 1 \) or, by rearranging

\[
0 = (z^k V_{M_z}(z, s), S(z)C(z)[\hat{H}(z) - \hat{H}(z, s)]).
\]

for \( k = 1, 2, \ldots, M \),

provided such a choice for \( s \) exists.

That such a choice indeed exists will be deduced by examining the limiting behavior of \( V_{M_z}(z, s) \) and \( \hat{H}(z, s) \) as \( s \) reaches any boundary point of the stability domain (5). Let \( \partial D \) denote the set of boundary points (i.e., points where at least one coefficient \( s_k \) has reached unit magnitude), such that the closure of \( D \) from (5) becomes \( \bar{D} = \partial D \cup \partial D \). We shall say that \( s \) belongs to the \( k \)th subboundary \( \partial D_k \) if

\[
|s_i| \begin{cases} 
\leq 1, & i < k \\
= 1, & i = k \\
< 1, & i > k 
\end{cases}
\]

so that \( \partial D = \bigcup_k \partial D_k \). The zeros of both \( V_{M_z}(z, s) \) and \( \hat{H}(z, s) \) are constrained functions of the poles, and the following two lemmas claim that the constraints in question are sufficient to ensure stability of \( V_{M_z}(z, s) \) and \( \hat{H}(z, s) \) as any poles reach the unit circle.

**Lemma 3:** If \( s \in \partial D_k \), then \( V_{M_z}(z, s) \) reduces to a stable and causal all-pass function of degree \( M - k \), parameterized by \( s_k = \pm 1 \) and \( s_{k+1}, \ldots, s_M \), and is odd with respect to \( s \) along \( \partial D \)

\[
V_{M_z}(z, -s) = -V_{M_z}(z, s), \quad \text{for all } s \in \partial D.
\]

**Lemma 4:** Suppose the input spectral density function \( S(z) \) is bounded and nonzero for all \( |z| = 1 \) and that the compensation filter \( C(z) \) is strictly minimum phase. If \( s \in \partial D_k \), then \( \hat{H}(z, s) \) reduces to a stable and causal function of degree not exceeding \( M - k \), parameterized by \( s_k = \pm 1 \) and \( s_{k+1}, \ldots, s_M \), and is even with respect to \( s \) along \( \partial D \)

\[
\hat{H}(z, -s) = \hat{H}(z, s), \quad \text{for all } s \in \partial D.
\]

The proofs are deferred to Section IV. We now exploit the following Borsuk fixed point Theorem [14, p. 46].

**Theorem 5:** Let \( \bar{D} \) be a closed, bounded, symmetric, and convex subset of \( \mathbb{R}^M \), and let \( F \) be a continuous mapping from \( \bar{D} \) to \( \mathbb{R}^M \). If \( F \) is odd along the boundary, i.e.,

\[
F(-s) = -F(s), \quad \text{for all } s \in \partial D
\]

then \( F \) admits a fixed point in \( \bar{D} \), i.e., \( F(s_{*}) = s_{*} \), for some \( s_{*} \in \bar{D} \).

The same conditions on \( F \) also ensure the existence of a zero in \( \bar{D} \), by considering the continuous mapping \( G(s) = s - F(s) \) from \( \bar{D} \) to \( \mathbb{R}^M \); the condition (11) yields \( G(-s) = -G(s) \) for all \( s \in \partial D \), which implies \( G(s_{*}) = s_{*} \), for some \( s_{*} \in \bar{D} \), or \( F(s_{*}) = 0 \).

To apply this result, we set

\[
F(s) = \begin{bmatrix} z \\ z^2 \\ \vdots \\ z^M \end{bmatrix}, S(z)C(z)[\hat{H}(z) - \hat{H}(z, s)].
\]

Since inner products are linear functions of their arguments, Lemmas 3 and 4 imply easily that \( F(s) \) is odd along the boundary, i.e., \( F(-s) = -F(s) \) for all \( s \in \partial D \). Theorem 5 then implies \( F(s_{*}) = 0 \) for some \( s_{*} \in \bar{D} \), as desired in view of (10).

**Remark 2:** With \( s \), denoting a zero of \( F(s) \), the function \( \hat{H}(z, s_{*}) \) is the transfer function obtained at a stationary point. Although its existence is established, a closed-form solution for \( \hat{H}(z, s_{*}) \) is not obtained. In [17], however, we show that a constructive procedure can be developed when the input \( \{u(n)\} \) is white noise, using interpolation theory. A corresponding result for nonwhite inputs is still an open issue.

**Remark 3:** Depending on the unknown system \( \hat{H}(z) \), a stationary point may occur along the boundary \( \partial D \). In this case, Lemma 4 shows that \( \hat{H}(z, s_{*}) \) suffers pole-zero cancellations. As (1) and (2) depend only on external signals, the locations of any such pole-zero cancellations are theoretically indeterminate; see [8, pp. 541–543] for a simulation example of this phenomenon.

### IV. PROOFS OF LEMMAS

We examine the functions \( V_{M}(z, s) \) and \( \hat{H}(z, s) \) as \( s \rightarrow \partial D \). With \( V(z, s) = [V_0(z, s), \ldots, V_{M_z}(z, s)] \), we may use the Schur parameters (4) to express \( V(z, s) \) in terms of \( z \hat{V}(z, s) \); solving for \( \hat{V}(z, s) \) leads to

\[
\hat{V}(z, s) = (I - zF(s))^{-1}g(s)
\]

in which the terms appear elementwise (indexed from zero) as

\[
[F(s)]_{ij} = \begin{cases} 
-s_i s_{j+1} \sum_{i+1} c_i, & i \leq j < M \\
0, & j < i - 1 \text{ or } j = M 
\end{cases},
\]

with the conventions \( s_0 = +1, s_i = \sin \theta_i, \) and \( c_i = \cos \theta_i \). Since \( \{V(z, s), V(z, s)\} = I \) for all \( s \) (e.g., [12] and [8]), the pair \( [F(s), g(s)] \) is orthogonal

\[
F(s)[F(s)]^\dagger + g(s)[g(s)]^\dagger = I_{M+1}, \quad \text{for all } s \in \bar{D}.
\]

One may now check that, as \( s \rightarrow \partial D_k \), \( F(s) \) and \( g(s) \) tend smoothly to the forms

\[
F(s) = \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix}, \quad g(s) = \begin{bmatrix} o_k \\ g_2(s) \end{bmatrix}, \quad s \in \partial D_k
\]

where \( F_1(s) \) has dimensions \( k \times k \), and \( F_2(s) \) and \( g_2(s) \) depend only on \( s_k = \pm 1 \) and \( s_{k+1}, \ldots, s_M \). Along the subboundary \( \partial D_k \), the Lyapunov equation (14) decouples into two equations, namely

\[
F_1 F_1^\dagger = I_k, \quad F_2 F_2^\dagger + g_2 g_2^\dagger = I_{M + 1 - k}.
\]

The first equality shows that \( F_1 \) is orthogonal and hence contains \( k \) eigenvalues on the unit circle, corresponding to the \( k \) unit-circle zeros of \( A_{M_z}(z) \) when \( s \in \partial D_k \) (e.g., [15]). The term \( F_2 \) must then contain the eigenvalues inside the unit circle, and the second equality shows that \((F_2, g_2)\) is a stable controllable pair.
Now, along $\partial D_k$, the final $M - k + 1$ entries $V_{k}(z, s), \ldots, V_{M}(z, s)$ (12) reduce to $(I - F_{2}(s))^{-2}g(s)$. One may check from (13) that $F_{2}(-s) = F_{2}(s)$ and $g_{2}(-s) = -g_{2}(s)$ for all $s \in \partial D_k$. This implies that, for all $s \in \partial D_k$

$$V_{i}(z, -s) = -V_{i}(z, s), \quad i = k, k + 1, \ldots, M. \quad (16)$$

Since the passage from $F$ to $F_2$ removes $k$ states, each $V_{i}(z, s)$, for $i \geq k$, drops in degree from $M$ to $M - k$. The choice $i = M$ in (16) gives Lemma 3.

We now examine the linear combination $\hat{H}(z, s) = \sum_{k} \nu_{k} V_{k}(z, s)$, and the behavior of the coefficients $\{\nu_{k}\}$ obtained from (8), as $s \to \partial D_k$. Suppose first that $s \in D$. From (12) we obtain

$$\langle V(z, s), z^{i} \rangle = \begin{cases} F_{1}^{i}, & i \geq 0 \\ 0, & i < 0 \end{cases}$$

and similarly, from (14)

$$\langle V(z, s), z^{i} V(z, s) \rangle = \begin{cases} F_{1}^{i}, & i \geq 0 \\ 0, & i = 0 \\ |F_{1}^{i}|, & i < 0 \end{cases}$$

If we introduce the series expansions

$$S(z)C(z)H(z) = \sum_{i=-\infty}^{\infty} p_{i} z^{i}$$

$$S(z)C(z) = \sum_{i=-\infty}^{\infty} q_{i} z^{i}$$

we may then write, for the terms in (8)

$$\langle V(z, s), S(z)C(z)H(z) \rangle = \sum_{i=-\infty}^{\infty} p_{i} \langle V(z, s), z^{i} \rangle$$

and, similarly

$$\langle V(z, s), S(z)C(z)V(z, s) \rangle = \sum_{i=-\infty}^{\infty} q_{i} \langle V(z, s), z^{i} V(z, s) \rangle$$

$$= q_{0}I + \sum_{i=1}^{\infty} q_{i}F_{1}^{i} + q_{-i}(\Lambda^{i})^{-i}$$

$$\Delta p(F) q(F)$$. Letting now $s \to \partial D_k$, we observe using (15) that

$$\lim_{s \to \partial D_k} \langle V(z, s), S(z)C(z)H(z) \rangle = \begin{bmatrix} 0_{k} \\ \frac{p(F_{1})}{q(F_{2})} g_{2} \end{bmatrix}$$

$$\lim_{s \to \partial D_k} \langle V(z, s), S(z)C(z)V(z, s) \rangle = \begin{bmatrix} q(F_{1}) \\ q(F_{2}) \end{bmatrix}$$. Using now the eigendecomposition $F_{1} = U \Lambda U^{t}$ for the orthogonal matrix $F_{1}$, where $\Lambda$ is diagonal and $U$ is unitary, we may diagonalize $q(F_{1})$ as

$$U^{t}q(F_{1})U = q_{0}I_{k} + \sum_{i=1}^{\infty} q_{i}(\Lambda^{i}) + q_{-i}(\Lambda^{i})^{-i}$$

$$= \begin{bmatrix} S(\lambda_{1})C(\lambda_{1}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & S(\lambda_{k})C(\lambda_{k}) \end{bmatrix}$$. Since $|\lambda_{1}| = \cdots = |\lambda_{k}| = 1$, the matrix $q(F_{1})$ is invertible provided that $S(z)C(z) \neq 0$ for all $|z| = 1$. If we now substitute the forms (17) into (8), we find that $\nu_{0} = \cdots = \nu_{k-1} = 0$ on $\partial D_k$. The function $\hat{H}(z, s)$ is then reduced to

$$\hat{H}(z, s) = \nu_{k}(s)V_{k}(z, s) + \cdots + \nu_{M}(s)V_{M}(z, s), \quad s \in \partial D_k$$

in which the remaining coefficients $\nu_{k}, \ldots, \nu_{M}$ are obtained from the subsystem

$$\begin{bmatrix} V_{k}(z, s) \\ \vdots \\ V_{M}(z, s) \end{bmatrix} = \begin{bmatrix} S(z)C(z)H(z) \\ \vdots \\ S(z)C(z)V_{M}(z, s) \end{bmatrix}$$

$$= \begin{bmatrix} \nu_{k}(s) \\ \vdots \\ \nu_{M}(s) \end{bmatrix} \quad (18)$$

The Appendix shows that $q(F_{2})$ is invertible whenever $C(z)$ is strictly minimum phase, and since $V_{i}(z, -s) = -V_{i}(z, s)$ for $s \in \partial D_k$, we obtain $\nu_{i}(s) = -\nu_{i}(s)$ for $s \in \partial D_k$. This gives $\hat{H}(z, -s) = \hat{H}(z, s)$ for all $s \in \partial D$. Since each $V_{i}(z, s)$, for $i \geq k$, has degree $M - k$ on $\partial D_k$, their linear combination $\hat{H}(z, s)$ has degree not exceeding $M - k$, to complete the proof of Lemma 4.

V. CONCLUDING REMARKS

Our main result shows that, in undermodeled cases, a stable transfer function exists which satisfies the equations for a mean stationary point of pseudo-linear regression algorithms. This is an improvement over other candidate methods for which stability of the resulting model in undermodeled cases has been shown only for certain classes of input sequences (e.g., [16]), or does not apply in general (e.g., [7]).

Whether an attractor point always exists is, with reference to [5], [6], still dependent on the SPR condition for choosing the compensation filter $C(z)$, involving now the denominator $A_{1}(z)$ of the model obtained at a stationary point. If the input $\{u(t)\}$ is white noise, the set of stationary points is invariant to the choice of $C(z)$ [8, p. 534]; there exists then a compensation filter compatible with the SPR condition. For colored inputs, by contrast, the denominator $A_{1}(z)$ obtained at a stationary point varies with the compensation filter $C(z)$, leading to a difficult problem of whether there exists a specific choice of $C(z)$ which renders $C(z)/A_{1}(z)$ SPR. This reflects a known shortcoming of this algorithm class: If the system to be identified is unknown, then so is the set of admissible compensation filters.

APPENDIX

Here we show the invertibility of the nonsymmetric matrix

$$q(F) = \begin{bmatrix} V_{k}(z, s) \\ \vdots \\ V_{M}(z, s) \end{bmatrix} S(z)C(z) \begin{bmatrix} V_{k}(z, s) \\ \vdots \\ V_{M}(z, s) \end{bmatrix} \quad (19)$$

whenever $C(z) = c_{1}z + \cdots + c_{N}z^{N}$ is strictly minimum phase, with $N$ an arbitrary integer. We assume $s \in D$ if $k = 0$ as in (8), or $s \in \partial D_k$ if $k > 0$ as in (18). We begin by writing

$$= \sum_{i=0}^{N} c_{i}R_{i}$$

$$(19) = \sum_{i=0}^{N} c_{i}R_{i}$$

in which $\{R_{0}, \ldots, R_{N}\}$ is a matrix autocorrelation sequence because the filters $V_{i}(z)$ are stable and $S(z)$ is a spectral density.
function. By stochastic realization theory (e.g., [18]), we may find an asymptotically stable realization of the form
\[ \mathbf{x}(n+1) = \mathcal{F} \mathbf{x}(n) + \mathbf{w}(n) \]
\[ \mathbf{y}(n) = \mathcal{F} \mathbf{w}(n) \]
with the property that if \( E[\mathbf{w}(n) \mathbf{w}^T(n)] = \delta_{n-m} \mathbf{I} \), then \( E[\mathbf{y}(n) \mathbf{y}^T(m)] = \mathcal{R}_{n-m} \) for \( 0 \leq n - m \leq N \). If this system is put in input normal form \( \{ \mathbf{I} = \mathcal{F} \mathcal{F}^T + \mathcal{G} \mathcal{G}^T \} \), a calculation shows that
\[ \mathcal{R}_i = E[\mathbf{y}(n+i) \mathbf{y}^T(n)] = \mathcal{F} \mathcal{F}^T \mathcal{H}^T, \quad i = 0, 1, \ldots, N. \]
The matrix (19) may then be written as
\[ (19) = \mathcal{H} \left( \sum_{i=0}^{N} c_i \mathcal{F}^T \right) \mathcal{H}^T. \]
Now, \( \mathcal{R}_0 = \mathcal{H} \mathcal{H}^T > 0 \) so that \( \mathcal{H} \) has full row rank. As such, (19) is singular only if \( \sum c_i \mathcal{F}^T \) is singular. If \( \{ \lambda_k \} \) are the eigenvalues of \( \mathcal{F} \), with \( |\lambda_k| < 1 \), those of \( \sum c_i \mathcal{F}^T \) become \( C(\lambda_k) \). But \( C(z) \neq 0 \) for all \( |z| \leq 1 \) if \( C(z) \) is strictly minimum phase so that (19) must be invertible.

**REFERENCES**


**A Unified Approach for Stability Analysis of a Class of Time-Varying Nonlinear Systems**

Chun-Bo Feng and Shumin Fei

**Abstract**—It is shown that an integral with negative feedback from its output terminal through a strictly passive system is still strictly passive. By using successive negative feedback around integrals, the conditions of strict passivity and a unified approach for constructing strictly passive systems are obtained for time-varying nonlinear systems of different orders. A new concept of “dissipation factor” is defined in this paper. The stability of a class of time-varying nonlinear systems is examined by using this passivity analysis.

**Index Terms**—Passive system, passivity analysis, stability analysis, time-varying nonlinear system.

I. INTRODUCTION

The stability analysis for general time-varying nonlinear systems is a difficult problem of long standing. Up to now the most important method for treating this problem is Lyapunov’s direct method. The solution of the stability problem by Lyapunov’s direct method depends on how the Lyapunov function is constructed, and the usefulness of this method is limited because there is no general method for constructing appropriate Lyapunov functions. The input–output (I/O) analysis of dynamic systems has drawn much attention since the 1970’s. The notion of “passivity” taking from the analogy to electric circuit has been widely used in analyzing the stability of a general class of dynamic systems, including time-varying nonlinear systems (see [1]). Stability analysis from the point of view of passivity has led to Lyapunov-theoretic counterparts of many results obtained from I/O point of view [2]–[4]. In [2] the conditions under which a nonlinear system can be rendered passive via smooth state feedback are studied for an affine time-invariant smooth nonlinear system. This problem is fully solved if there exists a \( C^2 \) storage function for the system. Yet how to search for the storage function or to determine the existence of a \( C^2 \) storage function is still unsolved. The coprime factorization of the causal operator and I/O stabilization with the state feedback have been studied by Sonntag [5] and [6] for a general time-invariant nonlinear system. Though the question of when a finite-dimensional time-invariant nonlinear system can be rendered passive via smooth state feedback has been solved, the question of how to design a strictly passive time-varying nonlinear system of different orders is still left open.

The passivity analysis of nonlinear systems is applicable not only to stability analysis, but also to network synthesis, robust analysis, and adaptive control of nonlinear systems [7], [8], and so on.

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