Reducing Collision Probability on a shared medium using a variational method

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ABSTRACT

We consider a network with N nodes competing for access to the channel using un-slotted ALOHA. When a request is sent, each node may answer after a certain backoff time. Only the first answer is of importance, that is why we want to minimize the loss rate of the first message. We derive the optimal backoff probability distribution which minimizes the collision probability on the first message answering to a request. Unlike previous works, we extract the collision probability in continuous time domain. To this goal, we use a variational method. This problem had only be solved before in a slotted context (i.e. discrete time domain), but we want to be able to manage later situations where the nodes are not perfectly synchronized, which requires to know how to solve it in the continuous time context.

Keywords

Medium access, ALOHA, random access, first message, collision probability, variational calculus

1. INTRODUCTION

In wireless networks, especially in sensor networks, situations often occur where only the first reply to a request is important. It is typically the case with election protocols (cf. [1] for example, or [2], [3]). Usually, an ALOHA mechanism is used to transmit the answers, and nodes derive their backoff times (according to certain probability distribution) before transmitting the answer. In order to optimize the performance of such protocols, we tried to identify the best distribution which minimizes the collision probability of the *first* answer that collides with the replies of the other nodes.

Several interesting works have been published until now. In [4], authors address exactly this issue in a context where the time is slotted. But we want to relax this assumption on the slotted synchronization between the nodes.This may be difficult to implement in some situations and a method is needed to deal with un-slotted situations. In [6] and [7], the same problem in the same slotted context is addressed, but the authors noticed that the collision probability can be reduced if the probability that the nodes answer is less than 1. An optimal value is given. Moreover, the solution is explicit in [7] while it is only recursive in [6]. In [5], the authors address the same question without assuming any synchronization, but they do not derive analytically an exact solution. Moreover, there is a mistake in the formula (7) of [5] where it is implicitely assumed a uniform distribution for the loss rate. We give the correct formula below: formula (7).

In this work, we identify the optimal backoff distribution mechanism in a non-slotted environment. The problem is then to find an optimal continuous function minimizing an integral. Then, we use the classical tools of the variational calculus to derive the solution.

2. MODELING ASSUMPTIONS AND FORMALIZATION OF THE PROBLEM

We reuse the same formalization as in [5]. We consider a node having N neighbors. It sends a query and each one of the other nodes sends a reply after a given backoff time. We are interested by the *first* answer. Collisions involving messages other than the first one is not considered to be a problem. Each node has a window of length D. We assume $t = 0$ at the beginning of the backoff period which corresponds to the receipt instant of the query by all the N nodes. Let $x_1, x_2, \cdots, x_N$ be the times when the expected answers of node $i \in [1; N]$ are sent. The duration of the messages is considered the same for all the messages and equal to d. Let x_{first} denote the minimum of the x_i: $x_{first} = \min_{i \in [1;N]} x_i$. The collision probability of the first message is denoted $P_{(D,N)}$ and is formally given by:

$$
\begin{aligned}
P_{(D,N)} &= P\left(\exists i \in [1; N],\right. \\
&\quad \left. x_i < x_{first} + d / x_i \neq x_{first}\right) \quad (1)
\end{aligned}
$$

Let $\forall x \in [0; D], y(x)$ be the density function of the backoff distribution.

3. CALCULATION OF $P_{(D,N)}$

Let us assume $N = 2$. When the first message is emitted at x_{first}, the probability that it collides with the second answer is:

$$
p_1 = \frac{\int_{x_{first}}^{x_{first}+d} y(x)dx}{\int_{x_{first}}^{D} y(x)dx} \quad (2)
$$

Then, with N independent nodes, this probability is, for

all x in $[0; D - d]$:

$$p_{N-1} = 1 - \left[1 - \frac{\int_{x_{first}}^{x_{first}+d} y(x)dx}{\int_{x_{first}}^{D} y(x)dx}\right]^{N-1}$$

$$= 1 - \left[\frac{\int_{x_{first}+d}^{D} y(x)dx}{\int_{x_{first}}^{D} y(x)dx}\right]^{N-1} \tag{3}$$

When x is in $[D - d; D]$, the collision probability is equal to 1. Thus, the mean loss rate for any x_{first} is:

$$P_{(D,N)} = \int_0^{D-d} \left(1 - \left[\frac{\int_{x_{first}+d}^{D} y(x)dx}{\int_{x_{first}}^{D} y(x)dx}\right]^{N-1}\right) y_{first}(x)dx$$

$$+ \int_{D-d}^{D} y_{first}(x)dx \tag{4}$$

where $y_{first}(x)$ is the distribution of x_{first}. Let us determine y_{first}. The cumulative distribution function of x_{first} is

$$P(x_{first} < x) = P\left(\min_{i \in [1;N]} x_i < x\right)$$

$$= 1 - P\left(\min_{i \in [1;N]} x_i \geq x\right)$$

$$= 1 - \prod_1^N (x_i \geq x)$$

$$= 1 - \left(\int_x^D y(u)du\right)^N \tag{5}$$

and thus, its density function is obtained by differentiation:

$$\frac{d}{dx}P(x_{first} < x) = Ny(x)\left(\int_x^D y(u)du\right)^{N-1} \tag{6}$$

The general formula for the mean collision probability is then:

$$P_{(D,N)} = \int_0^{D-d} \left[1 - \left(\frac{\int_{x_{first}+d}^{D} y(x)dx}{\int_{x_{first}}^{D} y(x)dx}\right)^{N-1}\right]$$

$$\times Ny(x_{first})\left[\int_{x_{first}}^{D} y(x)dx\right]^{N-1} dx_{first}$$

$$+ \int_{D-d}^{D} Ny(x_{first})\left[\int_{x_{first}}^{D} y(x)dx\right]^{N-1} dx_{first} \tag{7}$$

4. OPTIMAL DISTRIBUTION

Our problem is to find the optimal function y which minimizes $\Lambda(y) = P_{(D,N)}$. It is a typical variational problem, and we use the classical tools of the variational calculus. Let $y_\varepsilon(x) = y(x) + \varepsilon h(x)$ where h is any continuous function on $[0; D]$. If y achieves the minimum of $\Lambda(y)$, then $\left(\frac{d\Lambda(y_\varepsilon)}{d\varepsilon}\right)_{\varepsilon=0} = 0$. Then, by differentiating $\Lambda(y_\varepsilon)$ in order to obtain a condition on the optimal y and by using the fact that the fundamental lemma of the variational calculus stands that, for any function g such as $g(a) = g(b) =$

$0, \int_a^b f(u)g(u)du = 0$ implies $\forall u \in [a; b], f(u) = 0$, it can be schown that $\left(\frac{d\Lambda(y_\varepsilon)}{d\varepsilon}\right)_{\varepsilon=0} = 0$ implies:

$$\begin{cases} \forall x \in [(2k+1)d; 2(k+1)d], y(x) = 0 \\ \forall x \in [2kd; (2k+1)d], \\ \quad y(x) = y(x+2d)\left[\frac{\int_{x+2d}^{D} y(u)du}{\int_{2(k+1)d}^{D} y(u)du}\right]^{N-2} \end{cases} \tag{8}$$

$y(x)$ can thus be arbitrary chosen on the last interval of the type $[2kd; (2k+1)d]$ of the $[0; D]$ interval as long as $\forall x \in [0; D], y(x) \geq 0$ and $\int_0^D y(u)du = 1$.

5. COLLISION RATE FOR THE OPTIMAL DISTRIBUTION

From now, without loss of generality, it is assumed $\exists n \in \mathbb{N}; D = (2n+1)d$. Let us denote $z_k = \int_{2kd}^{(2k+1)d} y(u)du$. Only taking into account that $x \in [(2k+1)d; 2(k+1)d], y(x) = 0$, it can be easily shown from (7) that

$$P(D, N) = 1 - N \sum_{k=0}^{\frac{D-d}{2d}-1} z_k \left[\sum_{i=k+1}^{\frac{D-d}{2d}-1} z_i\right]^{N-1} \tag{9}$$

To inject the optimal distribution in (9) is exactly to consider in the calculation the property $\forall x \in [2kd; (2k+1)d], y(x) = y(x+2d)\left[\frac{\int_{x+2d}^{D} y(u)du}{\int_{2(k+1)d}^{D} y(u)du}\right]^{N-2}$, which gives by integration

$$z_k = \int_{2kd}^{(2k+1)d} y(x)dx$$

$$= \int_{2kd}^{(2k+1)d} y(x+2d)\left[\frac{\int_{x+2d}^{D} y(u)du}{\int_{2(k+1)d}^{D} y(u)du}\right]^{N-2} dx$$

$$= \frac{\left(\sum_{i=k+1}^{\frac{D-d}{2d}} z_i\right)^{N-1} - \left(\sum_{i=k+2}^{\frac{D-d}{2d}} z_i\right)^{N-1}}{(N-1)\left(\sum_{i=k+1}^{\frac{D-d}{2d}} z_i\right)^{N-2}} \tag{10}$$

Finally, injecting (10) into (9) leads to

$$P(D, N) = 1 - (1 - z_0)^{N-1} \tag{11}$$

6. DISCUSSION

The optimal distribution is defined by the recurrence equation (8). It is noteworthy to observe how the systematic method of the variational calculus leads automatically to a system avoiding frame overlaps as long as possible: emissions are only permitted each two intervals of length d (cf. Fig. 1). It does not mean that every frame must be sent at the beginning of such an interval. The emission of a frame is allowed at any time of an authorized interval. This is exactly the fundamental difference with the discrete case as studied in [6] or [4]. A particular attention must be paid to the choice of the initial distribution on the last non null

interval of length d: it must be well chosen in order to have the whole sum of the distribution on $[0; D]$ to be equal to 1.
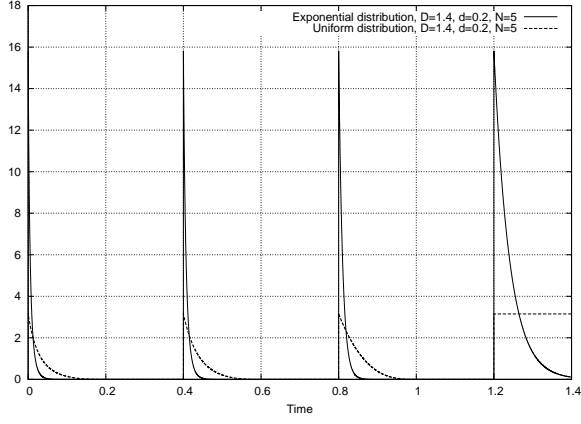
Figure 1: Optimal probability density functions

In figure 1 we display some examples of the shape of the optimal distributions of the probability functions for the cases where the distribution function on the last interval of length d inside the interval $[0; d]$ is uniform or exponential. In figure 2 several cumulative distributions functions are presented for the case where $D = 1.4$, $d = 0.2$, for different values for N. It can be observed that the higher N is, the later the convergence of the distribution toward 1.0 is. Some
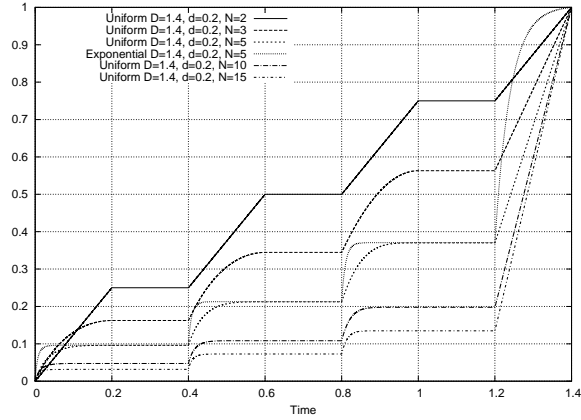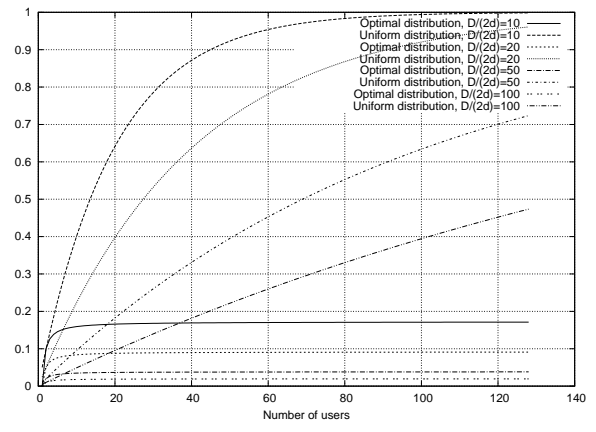

Figure 2: Optimal cumulative probability functions

example of collision rates are given in figure 3, together with a comparison with a simple uniform distribution.

In fact, (8) gives a condition for an optimal solution, but there may be several optima (possibly local optima). We presented one possible solution. It could be shown that the classical slotted solution satisfies also this condition. The main feature of the solutions is that the loss rate is bounded whatever the number of nodes. We are currently validating an explicit solution of the differential system (8).

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Figure 3: Examples of collision rates in function of the number of users