OPTIMIZING V ANTENNA ARRAYS USING A BAYESIAN DOA ESTIMATION CRITERION

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ABSTRACT

Experimentations have shown V-shaped uniform antenna arrays to be near-optimum for estimating the direction of arrival of a far-field source, whether the source position is fixed or random. We consider, as performance measure, the expected Cramer-Rao bound, normalized (for comparison purposes) to that of the commonly used uniform circular arrays. We study in details the behavior of V arrays for DOA estimation purposes. For large-sized V arrays, the performance measure shows a simple expression, enabling analytical solution of the subsequent array-geometry optimization problem. We obtain closed-form expressions of the shape and performance of optimal V arrays and learn about their ability to benefit from the available prior about the source direction.

1. INTRODUCTION

Direction finding is a major research field of statistical signal processing that has connections to a wide range of electronic systems and applications, civil and military. The direction of arrival (DOA) of an emitting source is estimated from the signals it induces at a set of sensors that form the antenna array. The most studied scenario is that of sensors displayed in a plane and a source anywhere in the array far-field.

The array geometry, i.e. sensors positions in the plane, has a notorious impact on DOA estimation, at least from two points-of-view. The first one is ambiguity, i.e. the ability to resolve sources with distinct DOAs. The second one is accuracy, i.e. the ability to achieve lower (mean square) errors. The first aspect can be secured by adopting uniform arrays for which adjacent sensors are separated by a constant (half-the-wavelength) spacing [1]. The second aspect, however, has remained largely unquantified, resulting into a very few theoretical results [2, 3] and heuristic-only geometry optimization techniques [4, 5]. Only recently has a simplification of the Cramer-Rao Bound (CRB) been obtained [6], telling a lot about the interaction between the array geometry and the estimation accuracy. It has been profitably used to determine the best antenna arrays, for instance, those that minimize the CRB [7, 8, 9]. The most noticeable outcome is that optimum uniform arrays show to have a shape close to V. This fact is true in both deterministic and probabilistic scenarios, i.e. if the source DOA is fixed (and unknown) or random (with known distribution). The V array is also convenient in practice because it is easy to manoeuvre (for instance to adapt the V angle to the available source prior, as we explain later).

The objective of this paper is to study in details the performance of V arrays in a scenario where the source DOA is not deterministic, but randomly located, following a known probabilistic distribution. It is reasonable to expect the available prior to help improving DOA estimation accuracy. In cellular communication networks, for example, we collect DOA statistics of base station users. The information is used to improve the quality of service and/or offer location-based services [10]. A variety of modeling techniques [11, 12] are used for this purpose.

For such a scenario, the CRB is no more a valid performance criterion. W.r.t. the theory of Bayesian estimation, Expected CRB (ECRB) is more relevant, and opportunistically, inherits the simple structure of the CRB. Consequently, analysis conducted in [7, 8] for fixed DOA, can be extended leading to analytical results which allow rich interpretation and are of great applicability. In particular, we give a detailed proof of an important result (presented without justification) in [9] about the performance of V arrays, followed by a detailed discussion and original illustrations.

The paper is organized as follows: First, in Sec. 2, we introduce the observation model and recall previous results. Second, in Sec. 3, we present the result of the optimization of V arrays. Finally, a conclusion is given in Sec. 4.

2. DATA MODEL AND PREVIOUS RESULTS

As illustrated by Fig. 1, a planar antenna array is composed of M identical and omni-directional sensors disposed, each, in the (x, y) plane. The position of the m-th sensor is given by its cartesian coordinates x_m and y_m, or, equivalently, by the complex number $\gamma_m \hat{=} x_m + iy_m$. A source is radiating a narrow-band signal characterized by a wavelength λ. For a source located in the antenna far-field, its DOA is given by a set of two angular parameters: the azimuth angle Φ and the elevation angle Θ, as shown in Fig. 1. Snapshots outputted

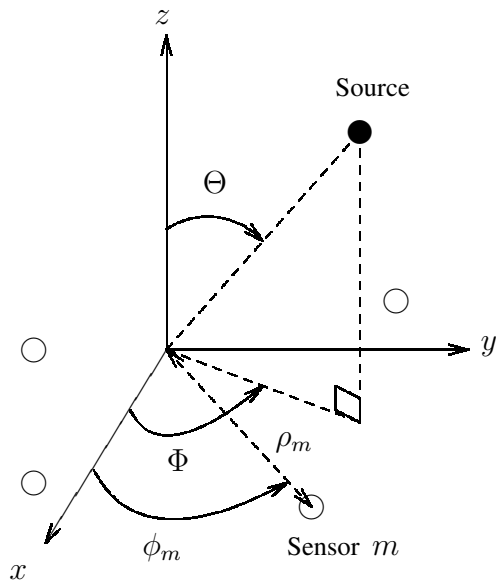

Fig. 1. Planar array and source DOAs.

by the M sensors are collected and used to infer about the the azimuth and elevation angles using a variety of algorithms. Some of them are capable of achieving the famous CRB.

We refer to the simple (but general) expression of the CRB obtained in [6], one that shows a simple sinusoidal function. The CRB is a relevant performance measure when the unknown (here azimuth and elevation angles) are deterministic parameters. In many applications, they are better modeled as randomly distributed parameters, with available a priori, in the form of a characteristic function $\varphi_\Phi(x) \hat{=} \mathrm{E}\left[\exp(j\Phi x)\right]$. In such a case, with reference to the theory of Bayesian estimation, the relevant criterion is the ECRB. We assume, as plausible in practice, that azimuth and elevation are independent. Then, the azimuth and elevation ECRBs are given by [9]

$$\begin{aligned}\bar{C}_{\Phi\Phi} &= A\,\mathrm{E}\left[\frac{1}{\sin^2(\Theta)}\right]\mathrm{E}\left[B\left(\Phi\right)\right]\\ \bar{C}_{\Theta,\Theta} &= A\,\mathrm{E}\left[\frac{1}{\cos^2(\Theta)}\right]\mathrm{E}\left[B\left(\Phi+\frac{\pi}{2}\right)\right],\end{aligned}$$

where only the function $B\left(\Phi\right)$ depends on the array geometry; other terms can be disposed of by normalizing the ECRB to that of a reference antenna, for instance, the M-sized d-spaced UCA. In deed, $B\left(\Phi\right) \hat{=} (S_0 + \Re\left[S_1 \exp(-2j\Phi)\right])/(S_0^2 - |S_1|^2)$ where S_0 and S_1 are functions of $\gamma_1, \cdots, \gamma_M$, the only to depend on the array geometry. For UCA, $B_{\mathrm{UCA}} \hat{=} B\left(\Phi\right) = 4\lambda^2 \sin^2(\pi/M)/(Md^2)$. Hence, we obtain the so-called normalized ECRBs [9]

$$\frac{\bar{C}_{\Phi\Phi}}{\bar{C}_{\Phi\Phi}|_{\mathrm{UCA}}} = \frac{1}{B_{\mathrm{UCA}}}\frac{S_0 + \Re\left[S_1\varphi_\Phi(-2)\right]}{S_0^2 - |S_1|^2}$$

$$\frac{\bar{C}_{\Theta\Theta}}{\bar{C}_{\Theta\Theta}|_{\mathrm{UCA}}} = \frac{1}{B_{\mathrm{UCA}}}\frac{S_0 - \Re\left[S_1\varphi_\Phi(-2)\right]}{S_0^2 - |S_1|^2}.$$

3. OPTIMIZATION OF LARGE SIZED V ARRAYS

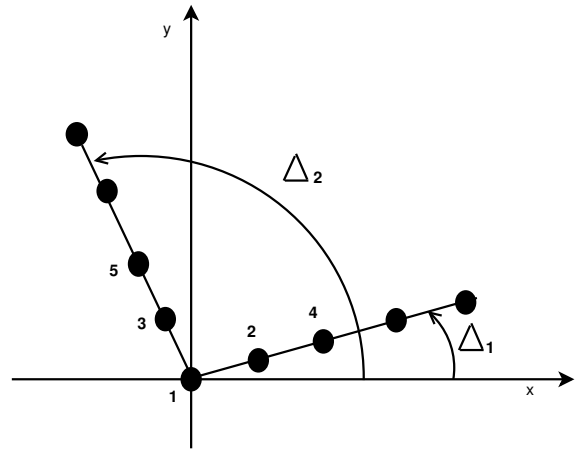

Fig. 2. The V-shaped antenna array.

As shown in Fig. 2, the V array is made of an odd number M of sensors. Sensor 1 is placed at the origin while, for $k = 1, \cdots, (M-1)/2$, sensors $2k$ and $2k+1$ are positioned such that $\gamma_{2k} = kd\exp(j\Delta_1)$ and $\gamma_{2k+1} = kd\exp(j\Delta_2)$. When M tends to infinity, it has been shown that [9] the normalized ECRBs become independent from M. For instance, they are equal to

$$\begin{aligned}r_\epsilon(\Delta_1,\Delta_2) \hat{=} &\frac{3}{2\pi^2}\frac{5 - 3\cos(\Delta_1 - \Delta_2)}{}\\ &\frac{+\epsilon\left[5\cos(\Delta_1-\Delta_2) - 3\right]\Re\left[\exp[j(\Delta_1+\Delta_2)]\varphi_\Phi(-2)\right]}{\sin^2(\Delta_2 - \Delta_1)}.\end{aligned}$$

where $\epsilon = 1$ (resp. -1) for the azimuth angle Φ (resp. elevation angle Θ). Our objective is to study in details functions $r_1(\Delta_1, \Delta_2)$ and $r_{-1}(\Delta_1, \Delta_2)$, and specifically, angles Δ_1 and Δ_2 for which the V array achieves a minimal $r_1(\Delta_1, \Delta_2)$ or $r_{-1}(\Delta_1, \Delta_2)$.

3.1. ECRB-Minimizing V arrays

We prove in Appendix A that $\min_{\Delta_1,\Delta_2} r_\epsilon(\Delta_1, \Delta_2)$

$$= \frac{3}{4\pi^2}\frac{(3-5\alpha)^2}{5 - 3\alpha - 4\sqrt{1-\alpha^2}}, \text{if } \alpha \neq \tfrac{3}{5} \quad (1)$$

$$= \frac{24}{5\pi^2}, \text{if } \alpha = \tfrac{3}{5}. \quad (2)$$

where the key-parameter $\alpha \hat{=} |\varphi_\Phi(2)|$ expresses the amount of prior available about the source azimuth angle. The proof of

(1) is detailed in Sec. A, while (2) is obtained by continuity of (1).

In Appendix A, we also prove that $r_\epsilon(\Delta_1^\dagger, \Delta_2^\dagger) = \min_{\Delta_1,\Delta_2} r_\epsilon(\Delta_1, \Delta_2)$ iff

$$\cos(\Delta_1^\dagger - \Delta_2^\dagger) = \frac{4\sqrt{1-\alpha^2}+3\alpha-5}{5\alpha-3} \quad (3)$$

$$\cos(\Delta_1^\dagger + \Delta_2^\dagger) = \epsilon\frac{\mathrm{E}\left[\cos\left(2\Phi\right)\right]}{\alpha} \quad (4)$$

$$\sin(\Delta_1^\dagger + \Delta_2^\dagger) = \epsilon\frac{\mathrm{E}\left[\sin\left(2\Phi\right)\right]}{\alpha}. \quad (5)$$

At last, we prove that

$$r_{-\epsilon}(\Delta_1^\dagger, \Delta_2^\dagger) = \frac{3}{16\pi^2}\frac{5\alpha-3}{\sqrt{1-\alpha^2}} \times \left(3+5\alpha+\frac{(5+3\alpha)(3-5\alpha)}{3\alpha-5+4\sqrt{1-\alpha^2}}\right).$$

Proof Bearing in mind that $\alpha_{\epsilon,u} = \alpha$, and following steps similar to those that led to (7), we can prove that

$$r_{-\epsilon}(\delta_3, \delta_4) = \frac{3}{2\pi^2}\frac{5+3\alpha-(5\alpha+3)\cos(\delta_3)}{1-\cos^2(\delta_3)}.$$

Implementing (11) and (12), where we denote $\beta \hat{=} \beta_{\epsilon,u}$, we get $\frac{2\pi^2}{3} r_{-\epsilon}(\delta_3, \delta_4) = \frac{5+3\alpha}{2\sqrt{\beta^2-1}\left(-\sqrt{\beta^2-1}+\mathrm{sign}(5\alpha-3)\beta\right)} + \frac{3+5\alpha}{2\mathrm{sign}(5\alpha-3)\sqrt{\beta^2-1}}$. As a consequence,

$$\begin{aligned}
& \mathrm{sign}\left(5\alpha-3\right)\sqrt{\beta^2-1}\frac{4\pi^2}{3} r_{-\epsilon}(\delta_3, \delta_4) \\
= & \frac{5+3\alpha}{\beta - \mathrm{sign}\left(5\alpha-3\right)\sqrt{\beta^2-1}} + 3 + 5\alpha, \text{i.e.} \\
& \mathrm{sign}\left(5\alpha-3\right)\frac{\sqrt{1-\alpha^2}}{|5\alpha-3|}\frac{16\pi^2}{3} r_{-\epsilon}(\delta_3, \delta_4) \\
= & \frac{5+3\alpha}{\beta + \mathrm{sign}\left(3-5\alpha\right) 4\frac{\sqrt{1-\alpha^2}}{|3-5\alpha|}} + 3 + 5\alpha, \text{and finally} \\
& \frac{\sqrt{1-\alpha^2}}{5\alpha-3}\frac{16\pi^2}{3} r_{-\epsilon}(\delta_3, \delta_4) \\
= & \frac{5+3\alpha}{\frac{3\alpha-5}{3-5\alpha} + 4\frac{\sqrt{1-\alpha^2}}{3-5\alpha}} + 3 + 5\alpha \\
= & \frac{(5+3\alpha)(3-5\alpha)}{3\alpha-5+4\sqrt{1-\alpha^2}} + 3 + 5\alpha \quad \blacksquare
\end{aligned}$$

3.2. Interpretation

First, notice that the angle $\Delta_1^\dagger - \Delta_2^\dagger$ between the two branches of the optimal V array does not depend on the angle of interest (i.e. not on ϵ). It is shown in Fig. 3 as function of the available azimuth prior, as expressed by α. The angle (parameter) of interest (or also ϵ) affects the orientation of the (optimal) V array, which is not unique by the way. To identify V arrays that satisfy (3)-(5), we let $\Delta_3 \hat{=} \arccos\left[(4\sqrt{1-\alpha^2}+3\alpha-5)/(5\alpha-3)\right]$ and Δ_4 be the angle in $[0, 2\pi[$ whose cosine and sine are given by the right-hand sides in (4)-(5), respectively. Let's also define $\Delta_5 \hat{=} (\Delta_4 - \Delta_3)/2$ and $\Delta_6 \hat{=} (\Delta_3 + \Delta_4)/2$.

On one hand, Δ_4 is, by definition, in $[0, \pi)$. Given that $\Delta_1 + \Delta_2$ is in $[0, 4\pi)$, the system with (4) and (5) has two possible solutions for $\Delta_1 + \Delta_2$. If $\epsilon\mathrm{E}\left[\sin\left(2\Phi\right)\right]$ is positive, then $\Delta_1 + \Delta_2 = \Delta_4$. Otherwise, $\Delta_1 + \Delta_2 = \Delta_4 + 2\pi$. On the other hand, Δ_3 is, by definition, in $[0, \pi)$. Given that $\Delta_1 - \Delta_2$ is in $[-2\pi, 2\pi)$, based on (4), there are 4 possible values for $\Delta_1 - \Delta_2$. It is equal to either Δ_3, $-\Delta_3$, $\Delta_3 - 2\pi$ or $2\pi - \Delta_3$.

Hence, we have 8 possible cases for the tuple (Δ_1, Δ_2) which are: (Δ_6, Δ_5), (Δ_5, Δ_6), $(\Delta_6 - \pi, \Delta_5 + \pi)$, $(\Delta_5 + \pi, \Delta_6 - \pi)$, $(\Delta_6 + \pi, \Delta_5 + \pi)$, $(\Delta_5 + \pi, \Delta_6 + \pi)$, $(\Delta_6, \Delta_5 + 2\pi)$ and $(\Delta_5 + 2\pi, \Delta_6)$. By definition, Δ_5 and Δ_6 are in $[-\pi, \pi)$ and $[0, 2\pi)$, respectively. Hence, some of these solutions correspond to identical V arrays. Actually, among these V arrays, only two are different. They are (Δ_5, Δ_6) and $(\Delta_5 + \pi, \Delta_6 + \pi)$. These two V arrays are obtained one from the other by symmetry w.r.t. to the origin.

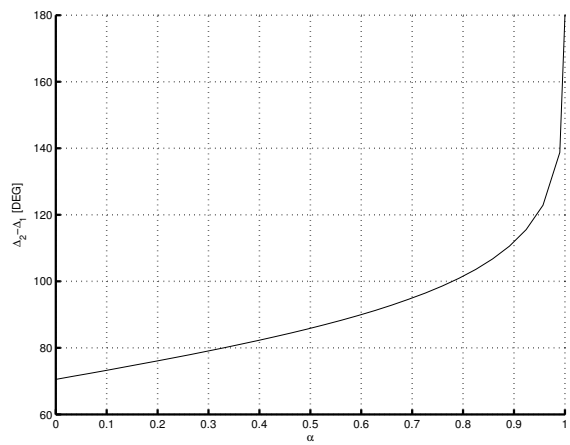

Fig. 3. The angle between the two branches of the optimal V array as function of parameter α.

3.3. Distributions of Φ such that $|\varphi_\Phi(2)| = 1$

The largest reduction of the ECRB is obtained for $|\varphi_\Phi(2)| = 1$. This is the case of a deterministic prior, but not only. More generally, we prove that α is equal to 1 iff the random azimuth has two possible values spaced by π, with arbitrary probabilities (hence, including the deterministic case). This is not surprising since the CRB is identical for two such directions. To establish the proof, let's, first, prove the following lemma.

Lemma Let I be a bounded interval and $f(x)$ be a summable complex-valued function over I. Then, $\left|\int_I f(x)dx\right| = \int_I |f(x)|\, dx$ iff $\arg\left[f(x)\right]$ is constant over I almost everywhere.

Proof Consider the constant $z \hat{=} \int_I f(x)dx$. Then

$$\begin{aligned}\left|\int_I f(x)dx\right| &= e^{-j\arg(z)}z \\ &= \Re\left(e^{-j\arg(z)}z\right) \\ &= \int_I \Re\left(e^{-j\arg(z)}f(x)\right)dx\end{aligned}$$

that is meant, here, to be equal to $\int_I |f(x)|\,dx$. However, $\Re\left(e^{-j\arg(z)}f(x)\right)$ is lower than its magnitude, i.e., lower than $|f(x)|$. Both being real-valued functions, their respective integrals over I can be equal iff the two functions are identical almost everywhere over I. By writing that $\Re\left(e^{-j\arg(z)}f(x)\right) = |f(x)|\Re\left(e^{j\arg[f(x)]-j\arg(z)}\right)$ $= |f(x)|\cos\{\arg[f(x)] - \arg(z)\}$ must be equal to $|f(x)|$, we conclude that $\arg[f(x)] - \arg(z)$ must be a constant (0 modulo 2π), where $\arg(z)$ is itself a constant. This terminates the proof.

In order to apply the above lemma, we rewrite $|\varphi_\Phi(2)| = 1$ as $\left|\int_{-\pi}^{\pi} f(\Phi)\exp(2j\Phi)d\Phi\right| = \int_{-\pi}^{\pi}|f(\Phi)\exp(2j\Phi)|d\Phi = 1$, where $f(\Phi)$ is the PDF associated with the random variable Φ. Hence, the argument of $f(\Phi)\exp(2j\Phi)$, i.e., $2j\Phi$ is a constant. Let's denote it Φ_0, chosen in $[-\pi, \pi)$. Then, Φ is equal to any number of the form $\Phi_0/2 + k\pi$, where k is any integer. There are two such numbers in $[-\pi, \pi)$ ■

4. CONCLUSION

We study in details the performance of uniform V arrays w.r.t. the estimation of the azimuth and elevation angles of a randomly located source. The expected CRB, normalized to that of the UCA, is our performance measure. It depends on the direction of the source (in fact, of the only azimuth angle) and the orientation of the branches of the V shape. Analytical expressions are obtained and closed-form solution of the optimal V array: its shape, orientation and accuracy.

A. PROOF OF (1)-(5)

We denote $\delta_3 \hat{=} \Delta_2 - \Delta_1$ and $\delta_4 \hat{=} \Delta_1 + \Delta_2$ and rewrite

$$r_\epsilon(\delta_3, \delta_4) \hat{=} \frac{3}{2\pi^2}\frac{5 - 3\cos(\delta_3) + \epsilon\left[5\cos(\delta_3) - 3\right]}{\sin^2(\delta_3)} \times \left\{\cos(\delta_4)\,\mathrm{E}\left[\cos(2\Phi)\right] + \sin(\delta_4)\,\mathrm{E}\left[\sin(2\Phi)\right]\right\}.$$

Minimization of r_ϵ w.r.t. δ_4 straightforwardly leads to

$$\tan(\delta_4) = \mathrm{E}\left[\sin(2\Phi)\right]/\mathrm{E}\left[\cos(2\Phi)\right]. \tag{6}$$

Thus, $\cos(\delta_4) = u\mathrm{E}\left[\cos(2\Phi)\right]/\alpha$, where $u = \pm 1$ and

$$\alpha \hat{=} \sqrt{\left\{\mathrm{E}\left[\sin(2\Phi)\right]\right\}^2 + \left\{\mathrm{E}\left[\cos(2\Phi)\right]\right\}^2},$$

i.e., $|\varphi_\Phi(-2)| = |\varphi_\Phi(2)|$, which, by definition, is less than one. Replacing δ_4 by its expression, we rewrite r_ϵ as follows

$$r_\epsilon(\delta_3, \delta_4) = \frac{3}{2\pi^2}\frac{5 - 3\cos(\delta_3) + \alpha_{\epsilon,u}\left[5\cos(\delta_3) - 3\right]}{\sin^2(\delta_3)},$$

where $\alpha_{\epsilon,u} \hat{=} u\epsilon\alpha$. This is a function of $\cos(\delta_3)$,

$$r_\epsilon(\delta_3, \delta_4) = \frac{3}{2\pi^2}\frac{5 - 3\alpha_{\epsilon,u} + (5\alpha_{\epsilon,u} - 3)\cos(\delta_3)}{1 - \cos^2(\delta_3)}. \tag{7}$$

The minimization of r_ϵ w.r.t. $\cos(\delta_3)$ leads to the second order equation $\cos^2(\delta_3) + 2\beta_{\epsilon,u}\cos(\delta_3) + 1 = 0$, where

$$\beta_{\epsilon,u} \hat{=} (5 - 3\alpha_{\epsilon,u})/(5\alpha_{\epsilon,u} - 3). \tag{8}$$

Notice that

$$\beta_{\epsilon,u}^2 - 1 = 16(1 - \alpha^2)/((3 - 5\alpha_{\epsilon,u})^2) \tag{9}$$

is always positive because α is in $[0, 1]$, by definition. Hence, $\cos(\delta_3)$ is equal to the one among $-\beta_{\epsilon,u} + \sqrt{\beta_{\epsilon,u}^2 - 1}$ and $-\beta_{\epsilon,u} - \sqrt{\beta_{\epsilon,u}^2 - 1}$ that has an absolute value lower than 1. Notice that the product of these two values is 1, so one and only one is a valid cosine. Now, one can verify from (8) that $\beta_{\epsilon,u} - 1 = 8\left(1 - \alpha_{\epsilon,u}\right)/\left(5\alpha_{\epsilon,u} - 3\right)$ and $\beta_{\epsilon,u} + 1 = 8\left(1 + \alpha_{\epsilon,u}\right)/\left(5\alpha_{\epsilon,u} - 3\right)$. Both have the same sign, since their respective numerators are positive.

On one hand, if $\alpha_{\epsilon,u} \geq 3/5$, then $\beta_{\epsilon,u} - 1$ and $\beta_{\epsilon,u} + 1$ are both positive. Hence, we have $-\beta_{\epsilon,u} \leq -1 \leq \beta_{\epsilon,u}$, implying that $(\beta_{\epsilon,u} - 1)^2 \leq \beta_{\epsilon,u}^2 - 1 \leq (\beta_{\epsilon,u} + 1)^2$, or also that $\beta_{\epsilon,u} - 1 \leq \sqrt{\beta_{\epsilon,u}^2 - 1} \leq \beta_{\epsilon,u} + 1$. Hence, $\cos(\delta_3) = -\beta_{\epsilon,u} + \sqrt{\beta_{\epsilon,u}^2 - 1}$. On the other hand, if $\alpha_{\epsilon,u} \leq 3/5$, then $\beta_{\epsilon,u} - 1$ and $\beta_{\epsilon,u} + 1$ are both negative. Hence, we have $\beta_{\epsilon,u} \leq -1 \leq -\beta_{\epsilon,u}$, implying that $(\beta_{\epsilon,u} + 1)^2 \leq \beta_{\epsilon,u}^2 - 1 \leq (\beta_{\epsilon,u} - 1)^2$, or also that $-\beta_{\epsilon,u} + 1 \leq \sqrt{\beta_{\epsilon,u}^2 - 1} \leq -\beta_{\epsilon,u} - 1$. Hence, $\cos(\delta_3) = -\beta_{\epsilon,u} - \sqrt{\beta_{\epsilon,u}^2 - 1}$. As a conclusion,

$$\cos(\delta_3) = \mathrm{sign}\left(5\alpha_{\epsilon,u} - 3\right)\sqrt{\beta_{\epsilon,u}^2 - 1} - \beta_{\epsilon,u}. \tag{10}$$

In order to update (7), we write that

$$\begin{aligned}\cos^2(\delta_3) &= 2\beta_{\epsilon,u}^2 - 1 - 2\mathrm{sign}\left(5\alpha_{\epsilon,u} - 3\right)\beta_{\epsilon,u}\sqrt{\beta_{\epsilon,u}^2 - 1} \\ \frac{\sin^2(\delta_3)}{2} &= 1 - \beta_{\epsilon,u}^2 \\ &\quad + \mathrm{sign}\left(5\alpha_{\epsilon,u} - 3\right)\beta_{\epsilon,u}\sqrt{\beta_{\epsilon,u}^2 - 1} \qquad (11) \\ &= \mathrm{sign}\left(5\alpha_{\epsilon,u} - 3\right)\sqrt{\beta_{\epsilon,u}^2 - 1} \\ &\quad \times\left[-\mathrm{sign}\left(5\alpha_{\epsilon,u} - 3\right)\sqrt{\beta_{\epsilon,u}^2 - 1} + \beta_{\epsilon,u}\right] \\ &= -\mathrm{sign}\left(5\alpha_{\epsilon,u} - 3\right)\sqrt{\beta_{\epsilon,u}^2 - 1}\cos(\delta_3), \qquad (12)\end{aligned}$$

so that we rewrite (7) as $\frac{2\pi^2}{3} r_\epsilon(\delta_3, \delta_4) =$

$$\frac{5 - 3\alpha_{\epsilon,u}}{2\sqrt{\beta_{\epsilon,u}^2 - 1}\left(-\sqrt{\beta_{\epsilon,u}^2 - 1} + \operatorname{sign}(5\alpha_{\epsilon,u} - 3)\,\beta_{\epsilon,u}\right)} + \frac{3 - 5\alpha_{\epsilon,u}}{2\operatorname{sign}(5\alpha_{\epsilon,u} - 3)\sqrt{\beta_{\epsilon,u}^2 - 1}}.$$

Using (8) and (9), we simplify the above as follows

$$\begin{aligned} & 2\sqrt{\beta_{\epsilon,u}^2 - 1}\frac{2\pi^2}{3} r_\epsilon(\delta_3, \delta_4) \\ = & \frac{5 - 3\alpha_{\epsilon,u}}{-4\frac{\sqrt{1-\alpha^2}}{|3-5\alpha_{\epsilon,u}|} + \frac{5-3\alpha_{\epsilon,u}}{|5\alpha_{\epsilon,u}-3|}} - |5\alpha_{\epsilon,u} - 3| \\ = & \frac{(5 - 3\alpha_{\epsilon,u})\,|5\alpha_{\epsilon,u} - 3|}{-4\sqrt{1-\alpha^2} + 5 - 3\alpha_{\epsilon,u}} - |5\alpha_{\epsilon,u} - 3|, \end{aligned}$$

so that $\frac{\sqrt{\beta_{\epsilon,u}^2-1}}{|5\alpha_{\epsilon,u}-3|}\frac{4\pi^2}{3} r_\epsilon(\delta_3, \delta_4) = \frac{5-3\alpha_{\epsilon,u}}{-4\sqrt{1-\alpha^2}+5-3\alpha_{\epsilon,u}} - 1$ or also $\frac{\sqrt{\beta_{\epsilon,u}^2-1}}{|5\alpha_{\epsilon,u}-3|}\frac{\pi^2}{3} r_\epsilon(\delta_3, \delta_4) = \frac{\sqrt{1-\alpha^2}}{-4\sqrt{1-\alpha^2}+5-3\alpha_{\epsilon,u}}$.
Using (9), $4\frac{\sqrt{1-\alpha^2}}{(5\alpha_{\epsilon,u}-3)^2}\frac{\pi^2}{3} r_\epsilon(\delta_3, \delta_4) = \frac{1}{\frac{5-3\alpha_{\epsilon,u}}{\sqrt{1-\alpha^2}}-4}$ and finally $16\pi^2\frac{\sqrt{1-\alpha^2}}{3} r_\epsilon(\delta_3, \delta_4) = \frac{(3-5\alpha_{\epsilon,u})^2}{\frac{5-3\alpha_{\epsilon,u}}{4\sqrt{1-\alpha^2}}-1}$.

To determine whether the minimum of r_ϵ is met with u equal to 1 or -1, we study the sign of $r_\epsilon(\delta_3, \delta_4)|_{u=1} - r_\epsilon(\delta_3, \delta_4)|_{u=-1}$, which, w.r.t. the above, is also that of

$$\frac{(3 - 5\alpha_{\epsilon,1})^2}{\frac{5-3\alpha_{\epsilon,1}}{4\sqrt{1-\alpha^2}} - 1} - \frac{(3 - 5\alpha_{\epsilon,-1})^2}{\frac{5-3\alpha_{\epsilon,-1}}{4\sqrt{1-\alpha^2}} - 1},$$

i.e., the same as the sign of $\frac{(3-5\epsilon\alpha)^2}{5-3\epsilon\alpha-4\sqrt{1-\alpha^2}} - \frac{(3+5\epsilon\alpha)^2}{5+3\epsilon\alpha-4\sqrt{1-\alpha^2}}$ which can be found to be equal to

$$\epsilon\alpha\frac{150\alpha^2 + 240\sqrt{1-\alpha^2} - 246}{25 - 9\alpha^2 - 40\sqrt{1-\alpha^2} + 16(1-\alpha^2)} = -6\epsilon\alpha.$$

As a conclusion, r_1 (resp. r_{-1}) is minimized by δ_3 and δ_4 for which $u = 1$ (resp. $u = -1$). Hence $u = \epsilon$ and $\alpha_{\epsilon,u} = \alpha$. For such δ_3 and δ_4 minimizing r_ϵ we have, on one hand, by application of (10),
$\cos(\delta_3) = \operatorname{sign}(5\alpha - 3)\sqrt{\beta_{\epsilon,\epsilon}^2 - 1} - \beta_{\epsilon,\epsilon}$. By means of (8) and (9), this can be proved to be equal to (3). On the other hand, $\cos(\delta_4) = \epsilon\mathrm{E}[\cos(2\Phi)]/\alpha$, in addition to (6) lead to (4) and (5).

The minimum achieved by r_ϵ is, independently from ϵ,

$$16\pi^2\frac{\sqrt{1-\alpha^2}}{3} r_\epsilon(\delta_3, \delta_4) = \frac{(3 - 5\alpha_{\epsilon,\epsilon})^2}{\frac{5-3\alpha_{\epsilon,\epsilon}}{4\sqrt{1-\alpha^2}} - 1} = \frac{(3 - 5\alpha)^2}{\frac{5-3\alpha}{4\sqrt{1-\alpha^2}} - 1},$$

which can be verified to be equal to (1).

B. REFERENCES

[1] T. Birinci and Y. Tanik, "Optimization of nonuniform array geometry for DOA estimation with the constraint on gross error probability," *Signal Processing*, vol. 87 (2007), pp. 2360-2369.

[2] Ü. Baysal and R. L. Moses, "On the geometry of isotropic arrays," *IEEE Trans. Signal Process.*, vol. 51, no. 6, pp. 1469-1478, Jun. 2003.

[3] M. Hawkes and A. Nehorai, "Effects of sensor placement on acoustic vector-sensor array performance," *IEEE J. Ocean. Eng.*, vol. 24, no. 1, pp. 33-40, Jan. 1999.

[4] Ü. Oktel and R. L. Moses, "A Bayesian approach to array geometry design," *IEEE Trans. Signal Process.*, vol. 53, no. 5, pp. 19191923, May 2005.

[5] P. J. Bevelacqua and C. A. Balanis, "Optimizing antenna array Geometry for interference suppression," *IEEE Trans. Antennas and Propag.*, vol. 55, no. 3, pp. 637-641, Mar. 2007.

[6] H. Gazzah and S. Marcos, "Cramer-Rao bounds for antenna array design," *IEEE Trans. Signal Process.*, vol. 54, no. 1, pp. 336-345, Jan. 2006.

[7] H. Gazzah and K. Abed-Meraim, "Optimum Ambiguity-Free Directional and Omni-Directional Planar Antenna Arrays for DOA Estimation," *IEEE Trans. Signal Processing*, vol. 57, no. 10, pp. 3942-3253, Oct. 2009.

[8] H. Gazzah, "Optimum Antenna Arrays for Isotropic Direction Finding," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 47, no. 2, pp. 1482-1489, April 2011.

[9] H. Gazzah and J.-P. Delmas, "Direction Finding Antenna Arrays for the Randomly Located Source," *IEEE Trans. Signal Process.*, accepted.

[10] S. Ahonen and P. Eskelinen, "Mobile terminal location for UMTS," *IEEE Aerosp. Electron. Syst. Mag.*, vol. 18, no. 2, pp. 23-27, Feb. 2003.

[11] K. I. Pedersen, P. E. Mogensen and B. H. Fleury, "A stochastic model of the temporal and azimuthal dispersion seen at the base station in outdoor propagation environments," *IEEE Trans. Veh. Technol.*, vol. 49, no. 2, pp. 437447, Mar. 2000.

[12] K. T. Wong, Y. I. Wu and M. Abdulla, "Landmobile radiowave multipaths DOA-distribution: Assessing geometric models by the open literatures empirical datasets," *IEEE Trans. Veh. Technol.*, vol. 58, no. 3, pp. 946958, Mar. 2010.